Time series analysis and examples

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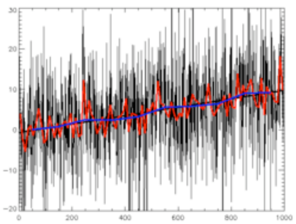
Outline

- Introduction
- Preprocessing (trend, seasonality)
- Frequency analysis for stationary signals
  - Model-free approach
  - Model-based approach
- Empirical Orthogonal Functions (EOF)
- Non stationary signals
  - Time-Frequency analysis
  - Time-Scale analysis
  - Wigner-Ville analysis
- Hidden Markov models

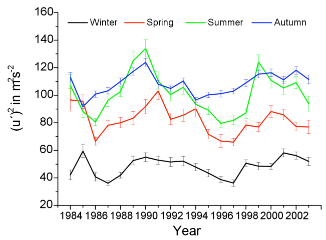
Introduction

Time series

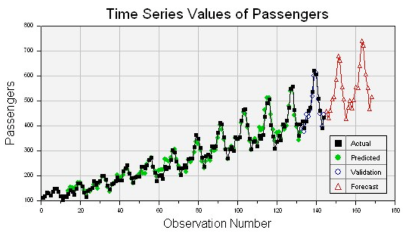
- Data obtained from consecutive measurements at equally spaced time intervals



http://www.statsoft.com/textbook/time-series-analysis



time series of seasonal mean gravity wave activity
http://www.itl.nist.gov/div898/handbook/pmc/section4/pmc41.htm



Monthly international passenger totals in 12 years
Box&Jenkins 1976

Extraction of information for

- Understanding of the system producing the data
- Forecasting, monotoring or control
- Identification of patterns in random signals

Time series analysis

- Model-free approach
- Model-based approach

- Time domain
- Frequency domain

Introduction

Applications

- Economic and sales Forecasting
- Stock Market Analysis
- Process and Quality Control
- Signal analysis (seismic, biological, speech, audio,..)

Examples in Environmental sciences

- Trends in temperature series
- Solar radiation measured by a pyranometer
- Wind speed time series
- Sea level records
- Internal gravity waves
- Seismic pattern dynamics
- Detection of cycles in atmospheric, geomagnetic and solar data

Definitions

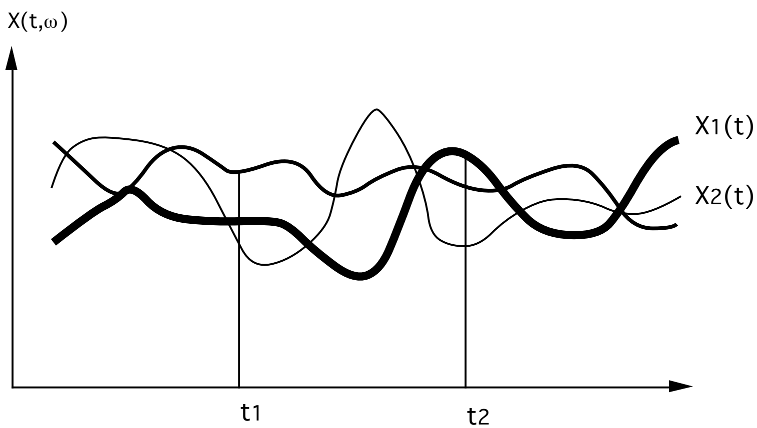
Time series and stochastic process

- Measured data : realizations of a more general stochastic process
- Stochastic process : random variable x(t) depending on time
- Time series : discretized stochastic process
  - sampling period : T
  - $T_k = kT$
  - Conventionaly T=1
- Univariate time series
  - Means a n-dimensional vector $\mathbf{x}=(x(1)\ x(2)\ \ldots\ldots\ldots\ x(n))^T$
- Multivariate time series
  - Each data is itself a m-dimensional vector (m simultaneous measurements)

    $$\mathbf{x}(k)=(x_1(k)\quad x_2(k)\quad \ldots\ldots\ldots x_m(k))^T$$
  - Means a nxm dimensional matrix $\mathbf{X} = (\mathbf{x}(1)\ \mathbf{x}(2)\ \ldots\ldots\ldots\ \mathbf{x}(n))^T$

Definitions

Time series and stochastic processes

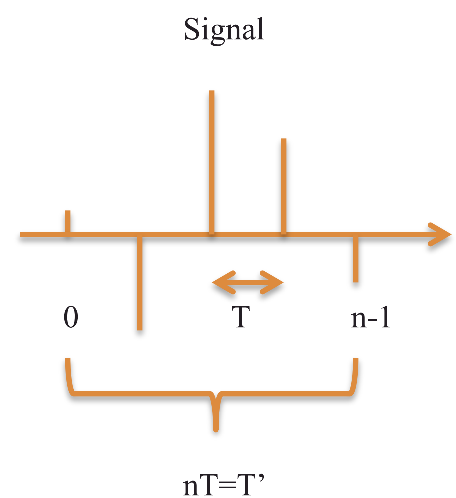


Continuous process (many realizations)

Sampling theorem

$$T \geq \frac{1}{2f_M}$$

f_M: maximal frequency



Time series

Stochastic analysis of time series

Real continuous random variable

- Probability density *p(x)*
- Moments $\mu_x^{(k)} = E\left[(x - E(x))^k\right] = \int_{-\infty}^{\infty} (x - E(x))^k p(x)\,dx$
  - Mean $m_x = E[x] = \int_{-\infty}^{\infty} xp(x)\,dx$
  - Variance $\sigma^2 = E[(x - E[x])^2] = \int (x - E[x])^2 p(x)\,dx$

Stochastic process

- Probability density *p(**x**)=p(x(1), x(2),……..,x(n))*
- Moments
  - Mean (order 1) $E(x(t_k)) = \int x(t_k) p(x(t_k))\,dx(t_k)$
  - Autocovariance (order 2)

$$\Gamma(x(t_k), x(t_l)) = \iint (x(t_k) - E(x(t_k)))(x(t_l) - E(x(t_l)))p(x(t_k), x(t_l))\,dx(t_k)\,dx(t_l)$$

  - Order k $E[(x(t_1) - E(x(t_1))(x(t_2) - E(x(2)) \ldots\ldots\ldots.(x(t_n) - E(x(t_n))]$

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Characteristic features

- Stationarity (first and second order)
  - Mean : independent of time k $m_y(k) = m_y \quad \forall k$
  - Variance : independent of k $\Gamma(0) = E[(x(k) - E[(x(k)])^2]$
  - Covariance : depends only on the interval $l - l' = k$

$$\Gamma(k) = E[\left(x(l) - E\left(x(l)\right)\right)\left(x(l+k) - E\left(x(l+k)\right)\right)]$$

- Linearity
  - Data are obtained by linear combination of white noise and past values (parametric model)

Characteristic features

- Trend
  - Increase or decrease of the mean → non stationary (order1)
- Seasonality
  - Periodically fluctuating pattern
  - Relationship with cycles of the real life (hour,day, month, year,..)
- Ergodicity
  - Equivalence between
    - statistical mean (E[ ])
    - Empirical mean $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}x(k)$ computed from a single realization on infinite observation duration T'=nT

Estimation of characteristic features (stationary case)

- Empirical mean

$$\hat{m} = \frac{1}{n}\sum_{k=1}^{n} x(k)$$

- Estimated variance

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{n} (x(k) - \widehat{m})^2$$

- Estimated covariance (or correlation with nul mean)

$$\hat{\Gamma}(k) = \frac{1}{n}\sum_{l=1}^{n} (x(l) - \hat{m})(x(l+k) - \hat{m})$$

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Preprocessing

- **Trend analysis and removal**
  - Smoothing for noise and outliers elimination
  - Fitting a function by least square $\min_{\theta} \sum_{k=1}^{n} e_{\theta}^{2}(k)$
    - Linear $e_{\theta}(k) = x(k) - (\alpha + \beta k)$ $\theta = (\alpha, \beta)$
    - Exponential $e_{\theta}(k) = x(k) - \exp(\alpha + \beta k)$ $\theta = (\alpha, \beta)$
    - Polynomial $e_{\theta}(k) = x(k) - (\alpha + \beta k + \gamma k^{2})$ $\theta = (\alpha, \beta, \gamma)$

Preprocessing

- **Seasonal analysis and removal**
  - Computation of Autocovariance in a specified range of lags
  - Detection of the periodicity between two peaks of autocovariance
  - Substraction (or division) of the seasonal component from the time series(depending of the additive or multiplicative model)

Frequency domain analysis

- **Change of Information representation system**

Time: x(t)=asin(2πft+φ) ***Frequency: a,f, φ***

  - Well fitted for oscillating or cyclic signals, but can be applied to any signal of finite energy.

- **Tool : Fourier Transform**

  - Signals are represented as a weighted sum of sine and cosine waves with different frequencies (inverse transform)

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

  - It gives the spectrum of the signal

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Frequency domain analysis

Application to stochastic processes

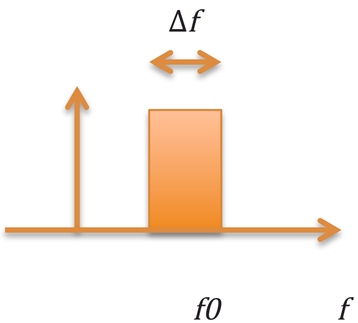
- The Fourier transform of a realization of a stochastic processes may not exist, because integration can diverge.
- Power spectral density : Fourier transform of the autocovariance

$$S(f) = \int \Gamma(\tau) e^{-j2\pi f \tau}\, d\tau$$

- Physical meaning

  – Instantaneous power $E(y_f^2(t))$

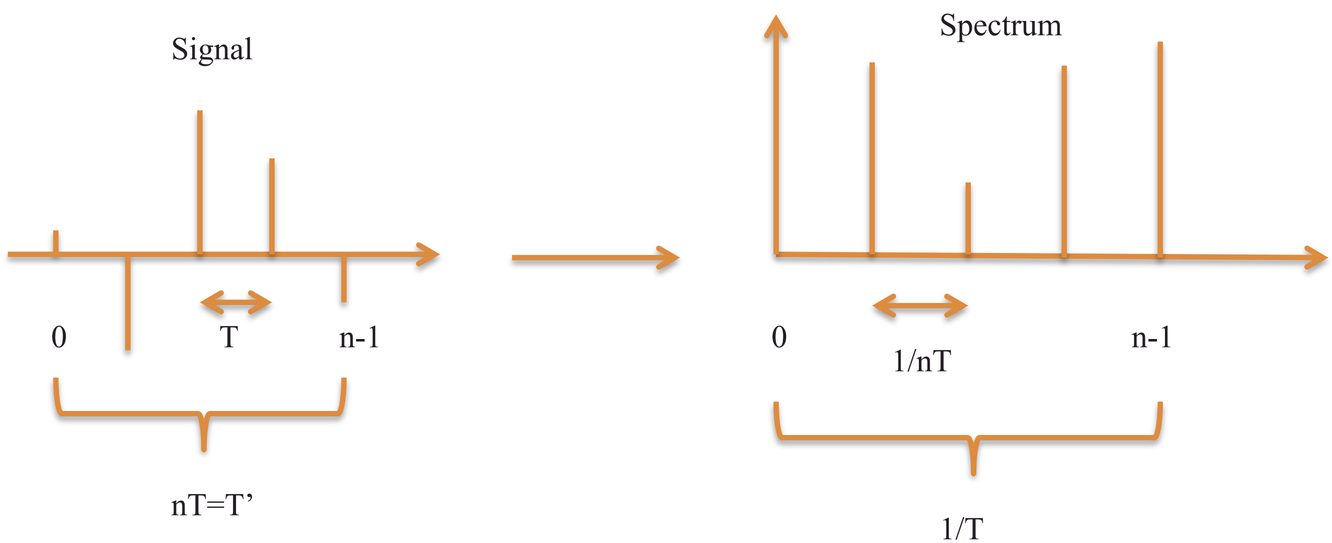
  – density $S_x(f) = \lim_{\Delta f \to 0} [\frac{1}{\Delta f} E(y_f^2(t))]$



Frequency domain analysis

Application to discretized signals

- Spectrum (Discrete Fourier Transform)



$$x(k) = \frac{1}{n}\sum_{l=0}^{n-1} X(l)e^{j\frac{2\pi kl}{n}}$$

$$X(l) = \sum_{k=0}^{n-1} x(k)e^{-j\frac{2\pi kl}{n}}$$

Frequency domain analysis

- **Power spectrum estimation of time series**
  - Applied to stationary time series (TS)
  - From a single realization (experiment)
  - With limited time record
- **Quality of estimation**
  - Resolution (depends on the TS length)
  - Variance of the estimator (accuracy)
  - Dynamic

⇒ Tradeoff between resolution and accuracy

Frequency domain analysis

- **True power spectrum** $$S_x(k) = \sum_{l=0}^{n-1} \Gamma(l) e^{-j\frac{2\pi lk}{n}} \qquad k = 0, \dots, n-1$$

- **Estimated power spectrum : Periodogram**
  - Two equivalent methods
    - Estimation of covariance and discrete Fourier transform of covariance

$$\hat{\Gamma}(k) = \frac{1}{n} \sum_{l=0}^{n-1} x(l)x(l+k) \qquad \hat{S}(k) = \sum_{l=0}^{n-1} \hat{\Gamma}(l) e^{-j\frac{2\pi lk}{n}} \qquad k = 0, \dots, n-1$$

    - Discrete Fourier transform of TS and square modulus

$$X(f) = \sum_{k=0}^{n-1} x(k) e^{-j2\pi kf} \qquad \hat{S}_x(f) = \frac{1}{n} |X(f)|^2$$

Frequency domain analysis

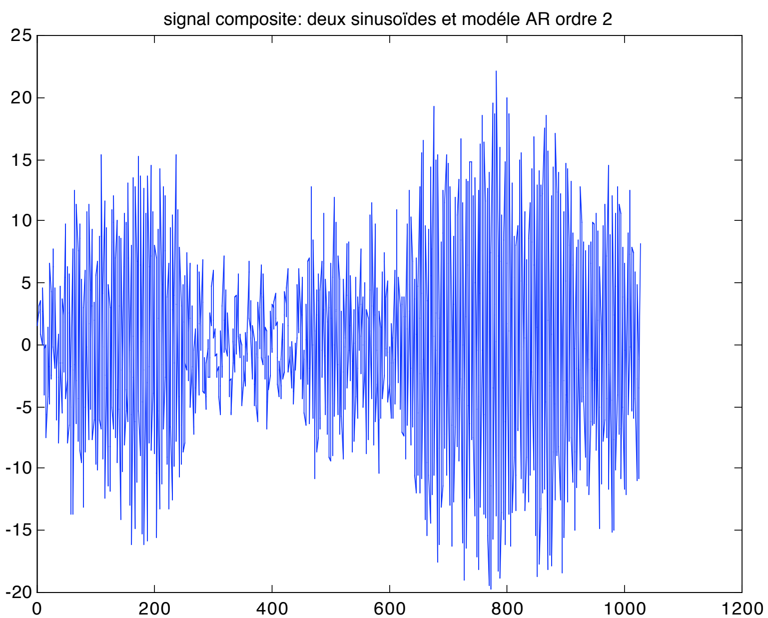
- **Periodogram : bad estimator**
  - Accuracy independent of the number n of data
  - Increase of n improves only resolution
- **Needs for accuracy improvement**
  - Averaging (Welch's method)
    - Division of TS in N frames
    - Computation of periodogram on each frame
    - Mean of periodograms
  - Filtering (Bartlett's method)
    - Discrete Fourier transform of windowed estimated correlation

⇒ Increase of accuracy, decrease of resolution

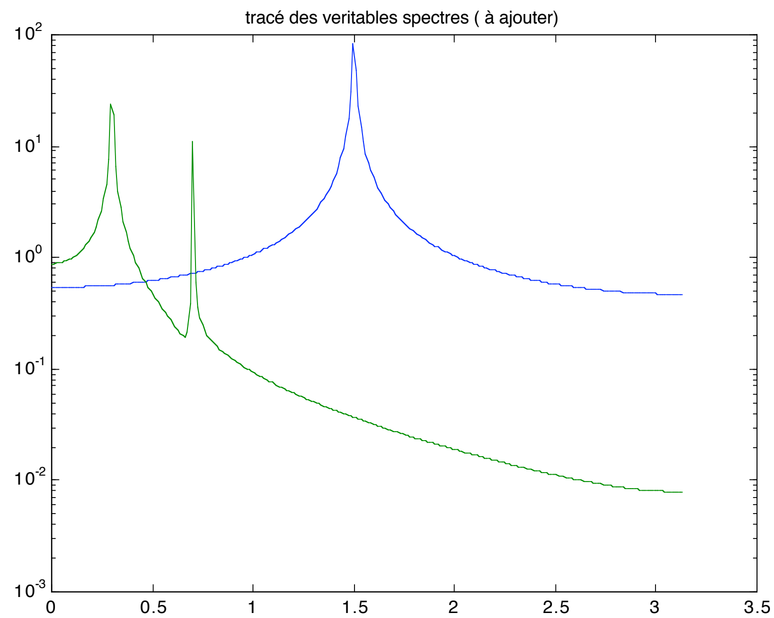
Frequency domain analysis

Example

- Complex signal as a sum of two sine waves and a stationary stochastic process with a damped correlation

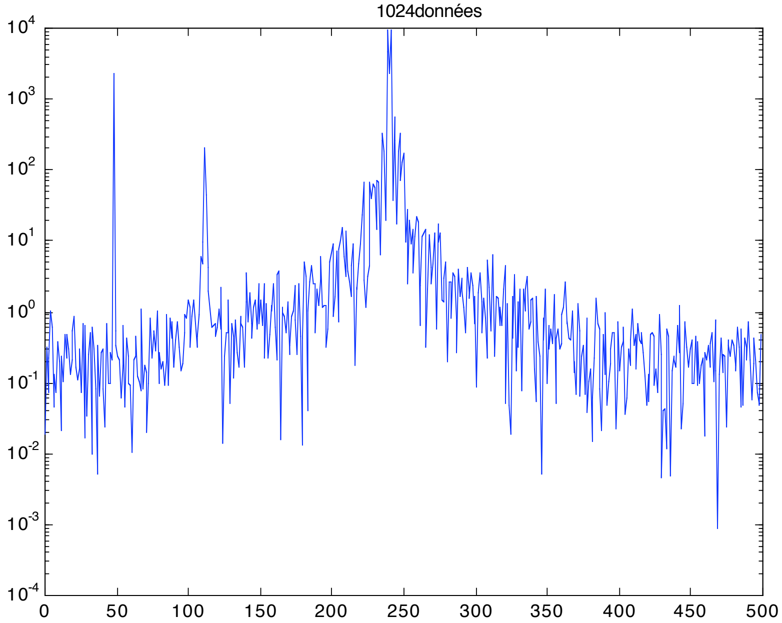


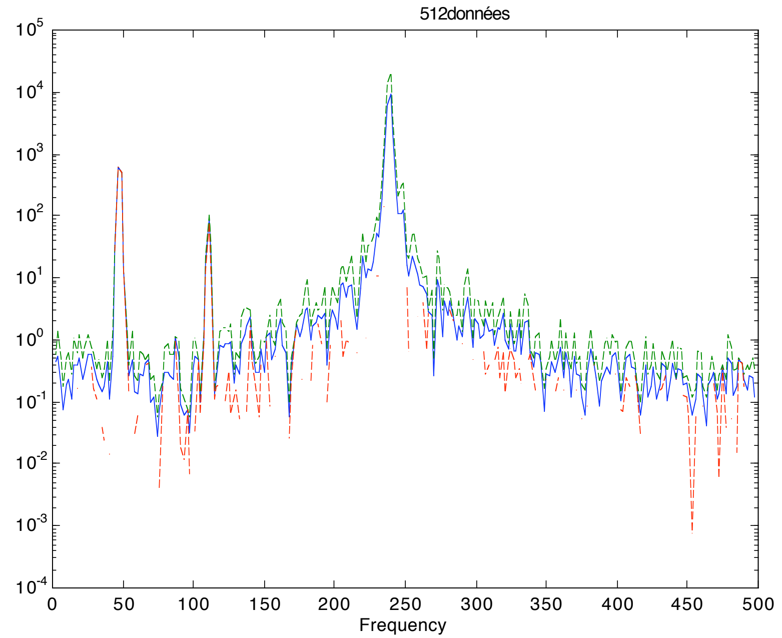
Signal



Theoretical power spectrum

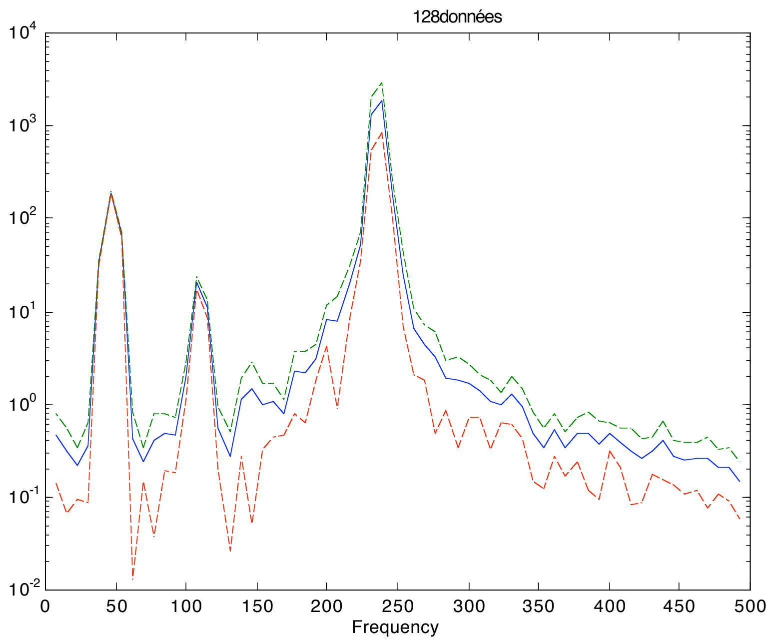
Frequency domain analysis

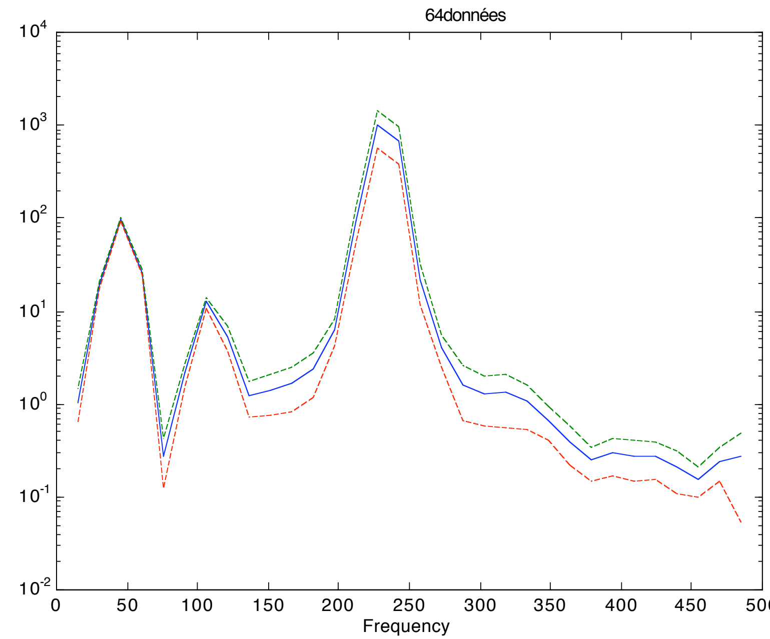




- Periodogram 1024 data
- Improved periodogram 512 data

Frequency domain analysis





- **Improved periodogram 128 data**
- **Improved periodogram 64 data**

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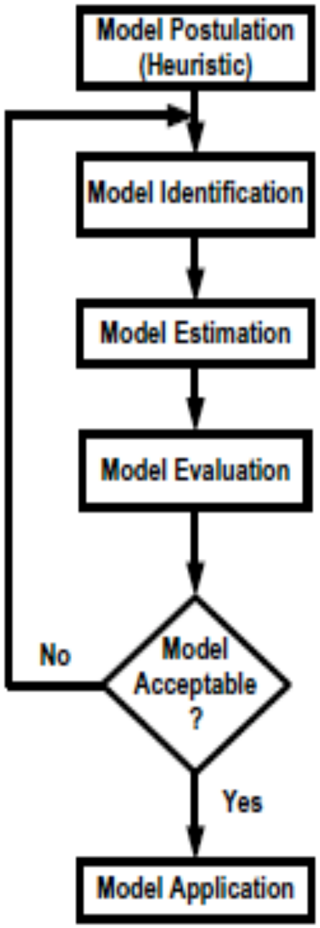
Frequency domain analysis

- Advantages
  - Usual
  - Show periodicity, cycles
  - Fits with stationary signals
- Drawbacks
  - Does not show discontinuities, change of frequencies → non stationary methods
  - Does not fit with short data length → model-based approach

Parametric model

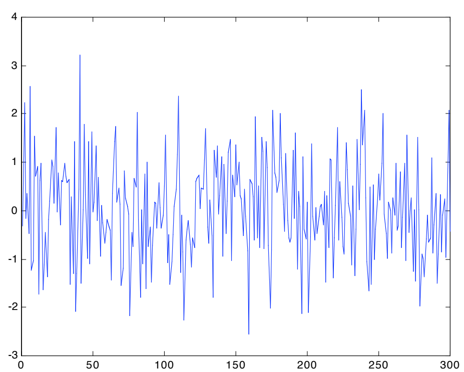
- **Model building** ***(Box and Jenkins, 1976)***
  - Model identification
    - number of model parameters
  - Model estimation
    - Estimation of the value of the parameters
  - Model validation
    - Checking of the model accuracy
  - Model forecasting
    - Use of the model to establish the confidence limits of the forecast



Parametric model

- **Simplest model for systems producing data**
  - Linear
  - Stationary

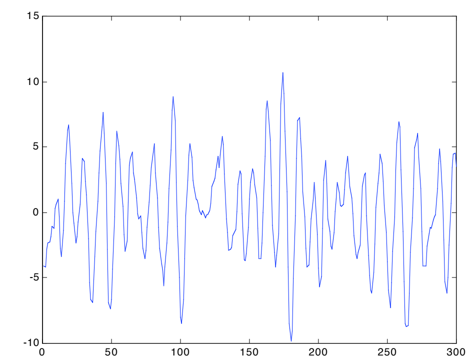
→ Filter definition



no correlation, no information

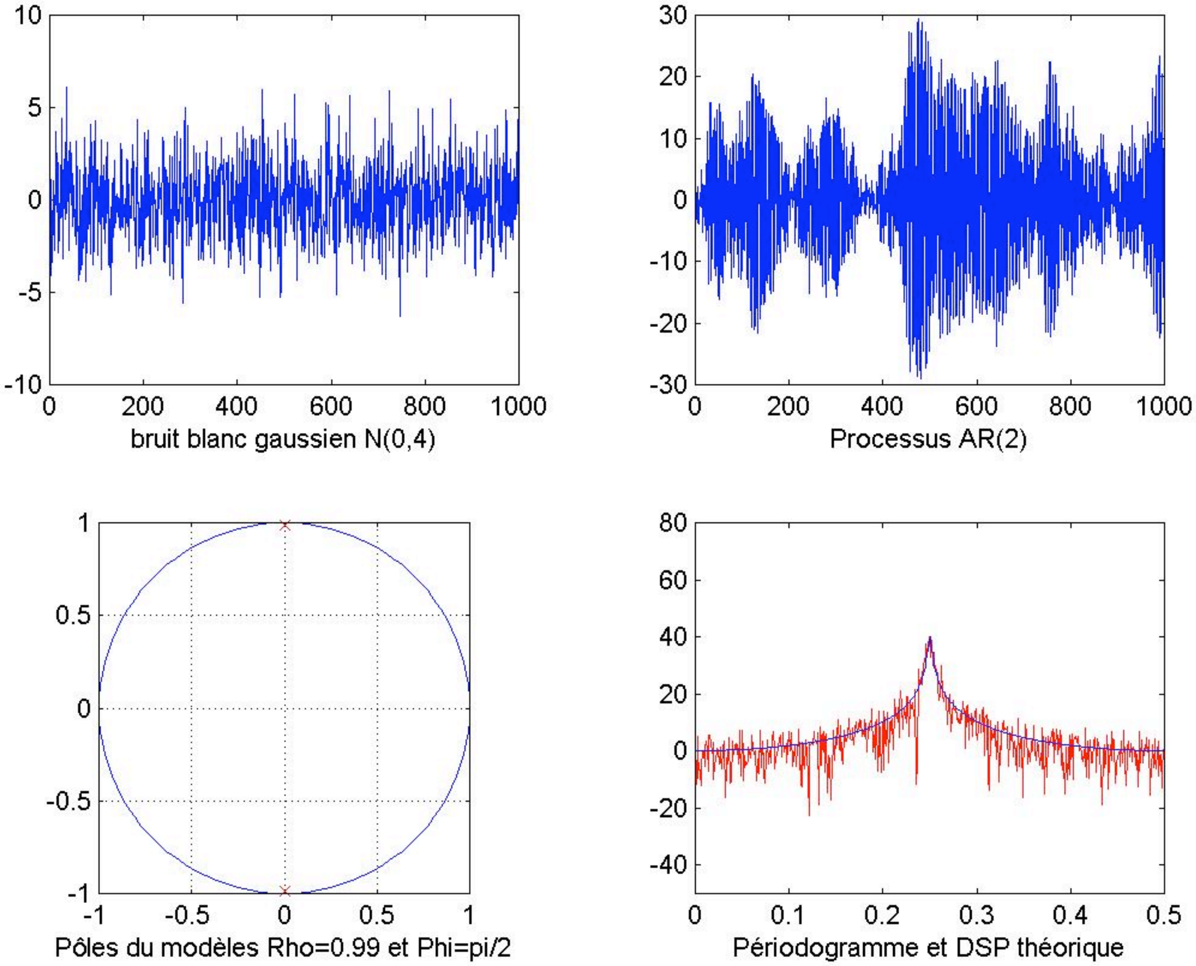
$$x(k) = -\sum_{i=1}^{n} a_i \, x(k-i) + \sum_{j=0}^{m} b_j n(k-j)$$

Model parameters: $(a_i, b_j \sigma_n^2)$?

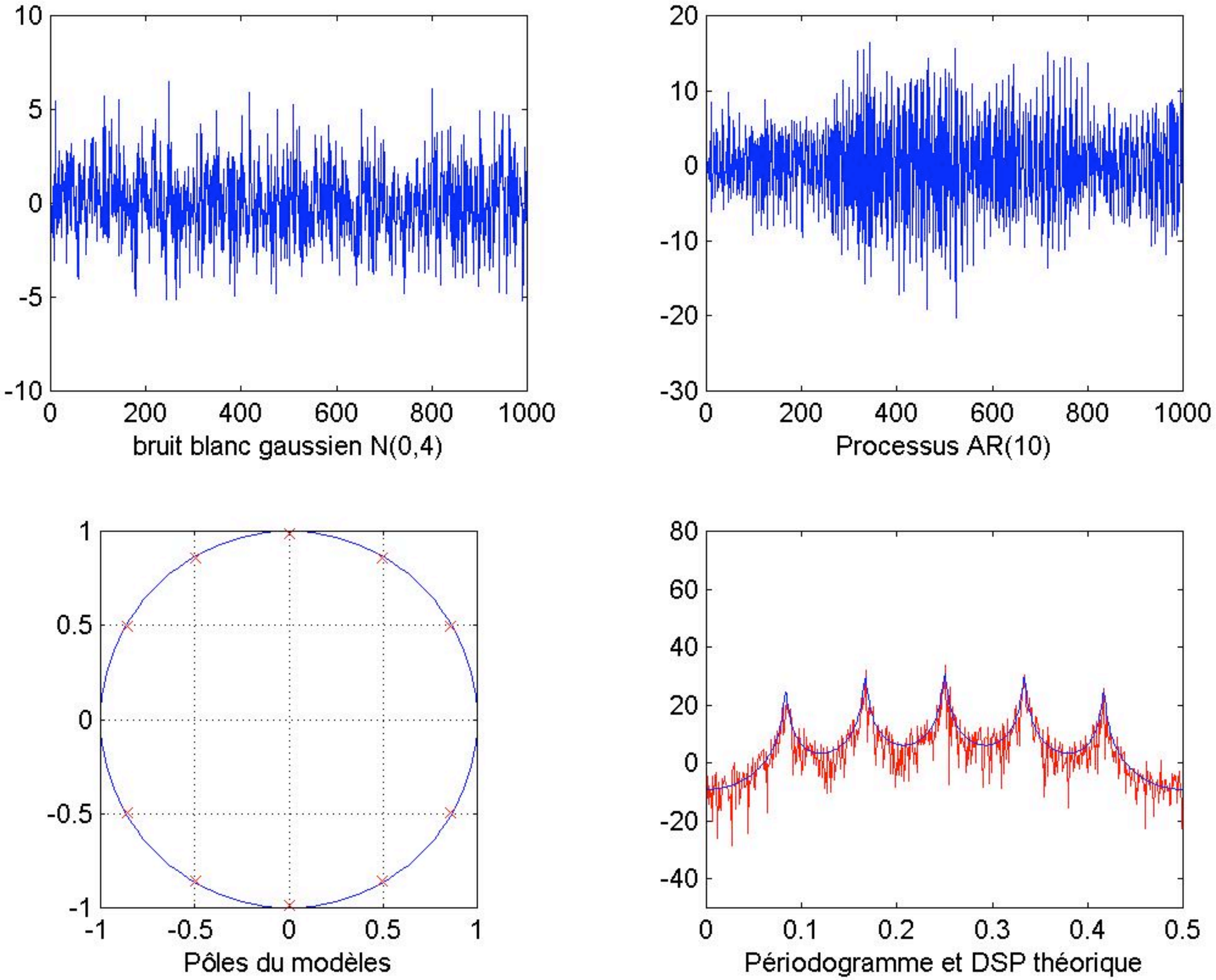


Correlation brought by the filter

Example of AR process

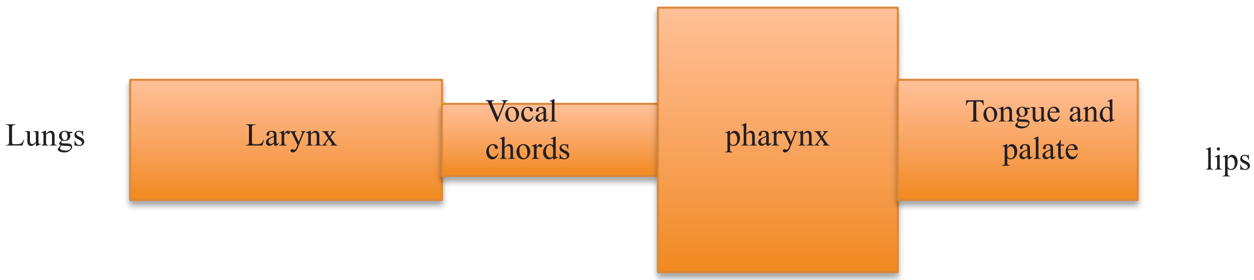


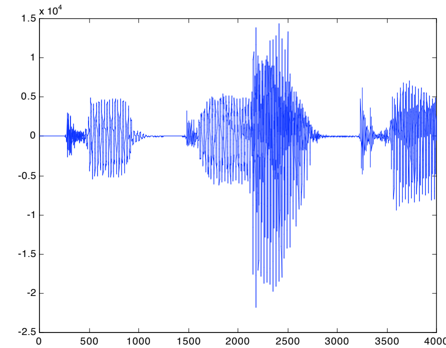
Example of AR process



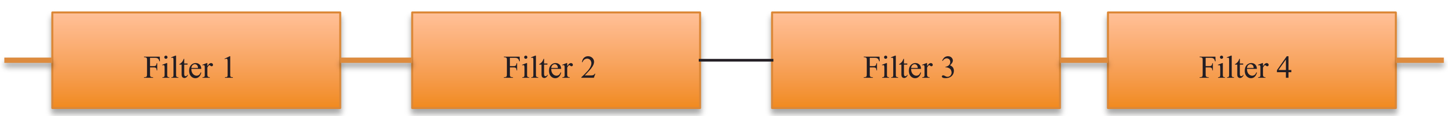
Regression models

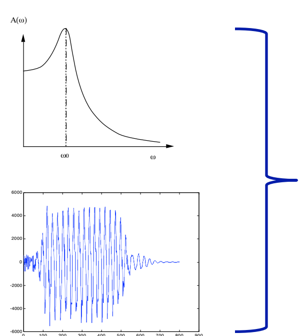
Example : simplified physical speech model

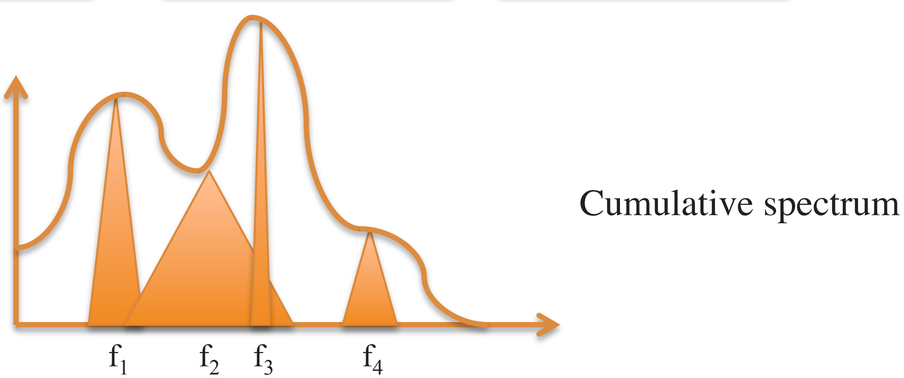




Cascade of 4 resonator filters







Parametric model

- **Autoregressive (AR) (all-poles filter)**

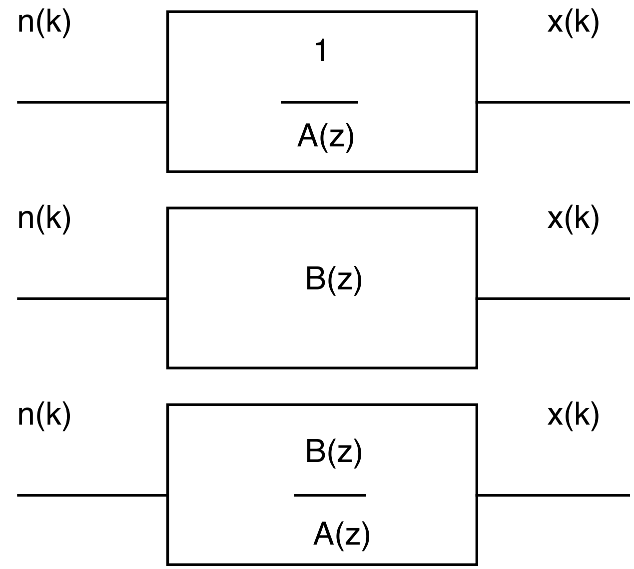
$$x(k) = \sum_{i=1}^{p} a_i \, x(k-i) + n(k)$$

- **Moving Average (MA) (all-zeros filter)**

$$x(k) = \sum_{j=0}^{q} b_j n(k-j)$$

- **ARMA (poles and zeros filter)**

$$x(k) = \sum_{i=1}^{p} a_i \, x(k-i) + \sum_{j=0}^{q} b_j n(k-j)$$



- **ARIMA (ARMA on** $\delta x(k) = x(k) - x(k-1)$ **)**

$$Z\big(x(k-i)\big) = z^{-i} Z(x(k))$$

  - Z: delay operator
  - n(k) : white noise with mean zero and correlation

$$\Gamma(0) = o^2$$

$$\Gamma(k) = 0 \quad k \neq 0$$

Model estimation

- AR model estimation by linear prediction

p centered data observed from (k-p) to (k-1) ⇨ linear prediction of the data value at time k

Predictive value

$$\hat{x}(k) = \sum_{i=1}^{p} a_i \, x(k-i)$$

Prediction error

$$e(k) = x(k) - \hat{x}(k) = x(k) - \sum_{i=1}^{p} a_i \, x(k-i)$$

Link with AR model

When is e(k) a white noise?

Model estimation

- Least square

$$\min_{a_i} \sum e^2(k)$$

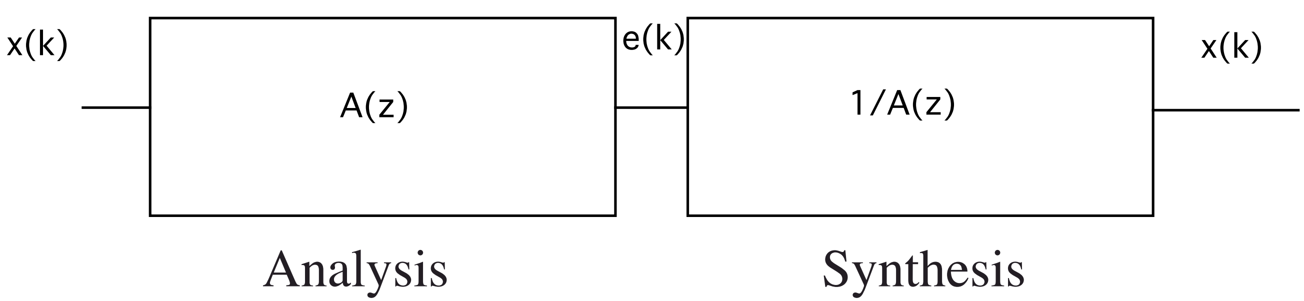
$$grad\left(\sum e^2(k)\right) = 0$$

$$\sum e(k)\frac{\partial e(k)}{\partial a_i} = 0 \qquad i = 1, \dots, p$$

$$\sum e(k)x(k-i) = 0 \qquad i = 1, \dots, p$$

  - ⇒ no correlation between error and data
  - ⇒ decorrelation of x(k) and e(k) tends to a white noise

- Inverse filtering



Model estimation

- Yule-Walker equations

$$\sum e(k)x(k-i) = 0 \qquad i = 1, \dots, p$$

$$\sum_{k=1}^{n} \left(x(k) - \sum_{j=1}^{p} a_i\, x(k-j)\right)x(k-i) = 0 \qquad i = 1, \dots, p$$

$$\sum_{j=1}^{p} a_i\, \hat{\Gamma}_x(i-j) = \hat{\Gamma}_x(i) \qquad i = 1, \dots, p$$

- Linear system of equations

$$\begin{vmatrix} \Gamma(0) & .. & .. & \Gamma(p-1) \\ . & . & & . \\ . & & . & . \\ \Gamma(p-1) & .. & .. & \Gamma(0) \end{vmatrix} \begin{vmatrix} a_1 \\ . \\ . \\ a_p \end{vmatrix} = \begin{vmatrix} \Gamma(1) \\ . \\ . \\ \Gamma(p) \end{vmatrix} \Rightarrow \begin{vmatrix} a_1 \\ . \\ . \\ a_p \end{vmatrix} = \begin{vmatrix} \Gamma(0) & .. & .. & \Gamma(p-1) \\ . & . & & . \\ . & & . & . \\ \Gamma(p-1) & .. & .. & \Gamma(0) \end{vmatrix}^{-1} \begin{vmatrix} \Gamma(1) \\ . \\ . \\ \Gamma(p) \end{vmatrix}$$

$$\sigma_e^2 = \sigma_x^2 - \sum_{i=1}^{p} a_i \Gamma(i)$$

Parametric model

Estimation of AR coefficients

- Estimation of covariance functions

$$\hat{\Gamma}(k) = \frac{1}{n} \sum_{l=0}^{n-1} x(l)x(l+k)$$

- Computation of coefficients and error variance by Yule-Walker equations

Associated power spectral estimate

$$S_x(f) = \frac{\sigma_e^2}{\left| \sum_{i=0}^{p} a_i \, e^{-j2\pi i f} \right|^2}$$

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Example of parametric models

- **AR 1**

$$x(k) = a_1 \mathrm{x(k\text{-}1)+n(k)}$$

$$a_1 = \frac{\Gamma(1)}{\Gamma(0)} \simeq \frac{\sum_{i=0}^{n-1} x(i)x(i+1)}{\sum_{i=0}^{n-1} x(i)^2}$$

$$\sigma_n^2 = \sigma_x^2 - a_1\Gamma(1) = \sigma_x^2 - \frac{\Gamma(1)^2}{\sigma_x^2}$$

- **AR 2**

$$x(k) = a_1 \mathrm{x(k\text{-}1)} + a_2 x(k-2) + \mathrm{n(k)}$$

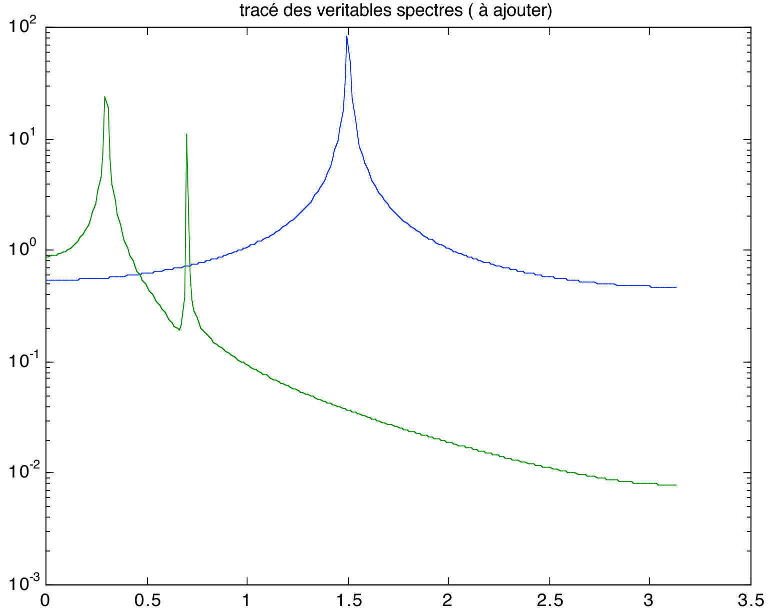
$$a_1 = \frac{\Gamma(1)\Gamma(0) - \Gamma(1)\Gamma(2)}{\Gamma(0)^2 - \Gamma(1)^2}$$

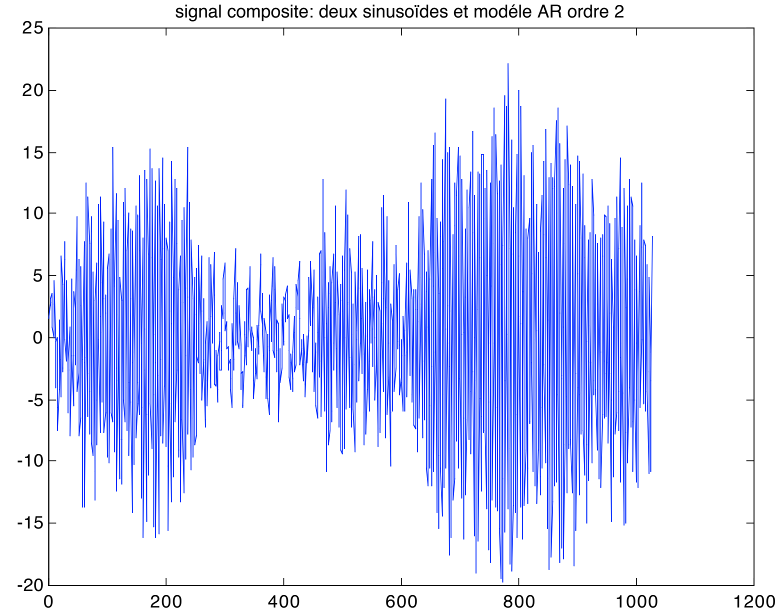
$$a_2 = \frac{\Gamma(1)^2 - \Gamma(0)\Gamma(2)}{\Gamma(0)^2 - \Gamma(1)^2}$$

$$\sigma_n^2 = \sigma_x^2 - a_1\Gamma(1) - a_2\Gamma(2)$$

Parametric model

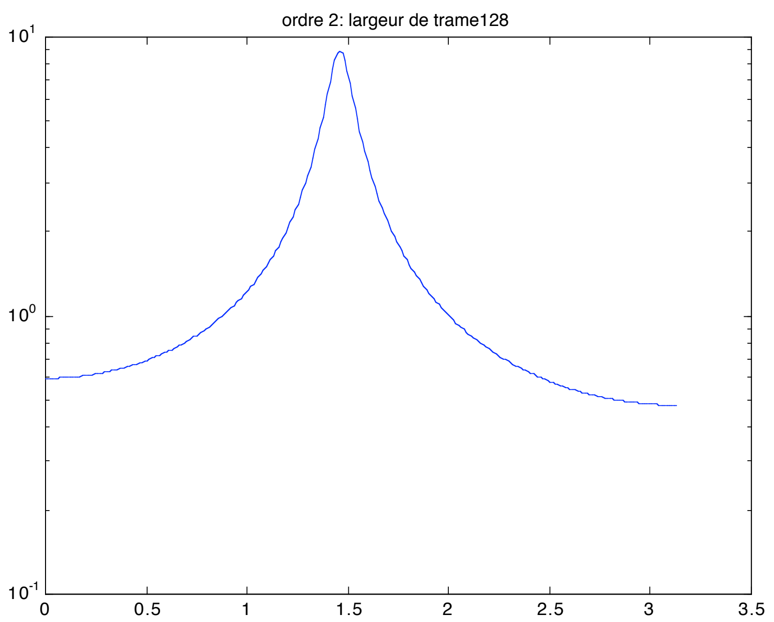
- Example
  - Complex signal as a sum of two sine waves and a stationary stochastic process with a damped correlation

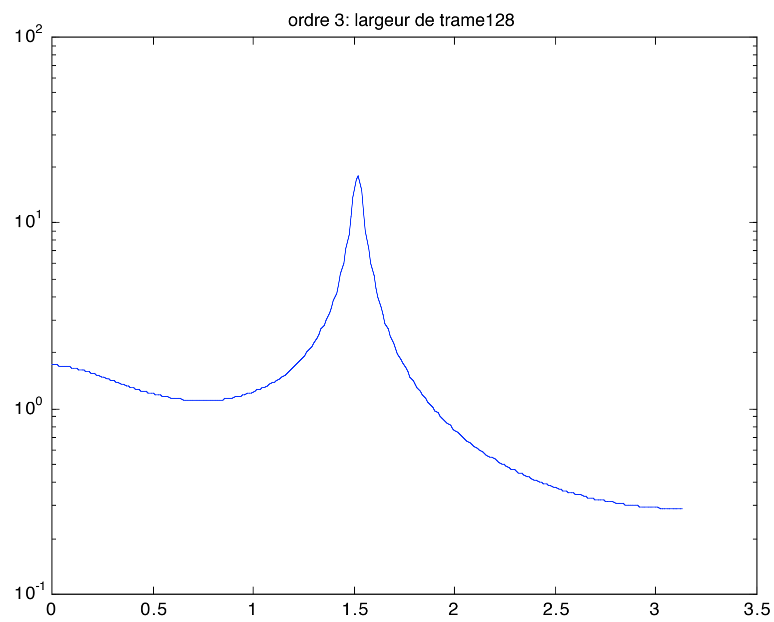




Parametric model

- AR model



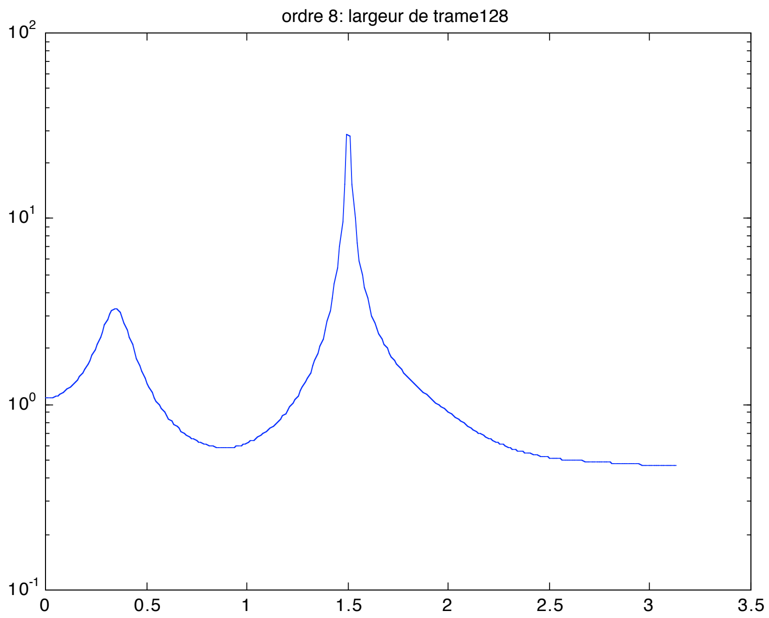


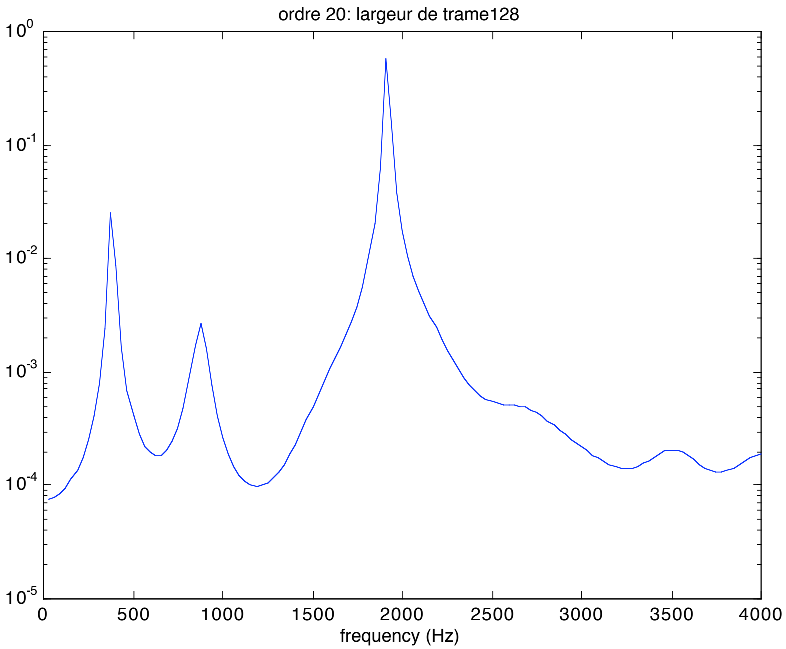
P=2, n=128

- **P=3,n=128**

Parametric model

- AR model





- **P= 8,n=128**

P=20,n=128

Model identification (order estimation)

- Minimization of a function of p given by information-theoretic approach
  - Rissanen or Bayesian Information Criterion (BIC) for gaussian data

$$RIS(p) = \ln\left(\sigma_e^2(p)\right) + p\frac{\ln(n)}{n}$$

  - Akaïke Information criterion (AIC)

$$AIC(p) = \ln\left(\sigma_e^2(p)\right) + \frac{2p}{n}$$

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Expansion on orthogonal basis functions

Advantages

- More compact representation of the signal energy
- Convenient for characterizing dominant pattern of variability

Time series

$$x(k) = \sum_{i=1}^{n} c_i \, f_i(k) \qquad c_i = \sum_{i=1}^{n} x(k) f_i^*(k)$$

- where $\sum_k f_i(k) f_j^*(k) = \delta_{ij}$ $\quad \delta_{ij} = 1 \;\; i = j, \delta_{ij} = 0 \;\; i \neq j$
- Complex exponentials are a special case and define a set of orthonormal functions

$$f_i(k) = \frac{1}{\sqrt{n}} e^{j\frac{2\pi i k}{n}}$$

- Fourier transform is also an expansion on orthonormal basis

Empirical Orthogonal Functions (EOF)

- **Coefficients must be uncorrelated random variables**

$$E\left[c_i c_j^*\right] = \lambda_i \delta_{ij}$$

- **Solution**

$$E\left[c_i c_j^*\right] = \sum_k \sum_l E[x(k)x(l)]\, f_i(k) f_j^*(l) = \lambda_i \delta_{ij}$$

  - as $\sum_k f_i(k)\, f_j^*(k) = \delta_{ij}$

$$\sum_k E[x(k)x(l)]\, f_i(k) = \lambda_i f_i(l)$$

Empirical Orthogonal Functions (EOF)

- Stationary case

$$\Gamma(k-l) = E[x(k)x(l)]$$

$$\sum_{k=1}^{n} \Gamma(k-l) f_i(k) = \lambda_i f_i\,(l) \qquad l = 1, \dots\dots .., n$$

$$\begin{bmatrix} \Gamma(0) & \cdots & \Gamma(n-1) \\ \vdots & \ddots & \vdots \\ \Gamma(n-1) & \cdots & \Gamma(0) \end{bmatrix} \begin{bmatrix} f_i(1) \\ \vdots \\ f_i(n) \end{bmatrix} = \lambda_i \begin{bmatrix} f_i(1) \\ \vdots \\ f_i(n) \end{bmatrix}$$

Solution : Eigenvectors and eigenvalues of the covariance matrix

$$\Gamma\Phi = \Lambda\Phi$$

$$\Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\Phi = [\Phi_1 \quad \cdots \quad \Phi_n]$$

$$\Phi_i = [f_1 \quad \cdots \quad f_n]^T$$

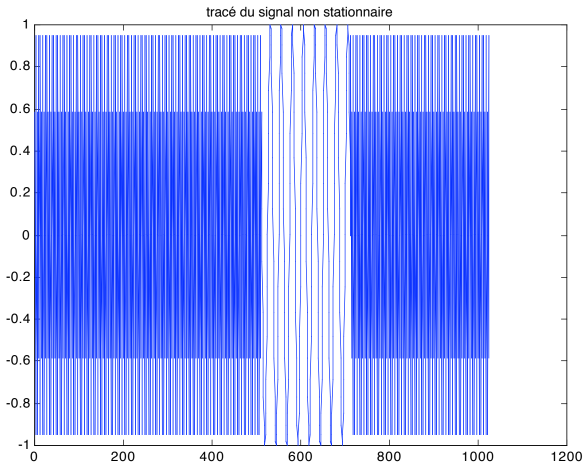
Other names for this method

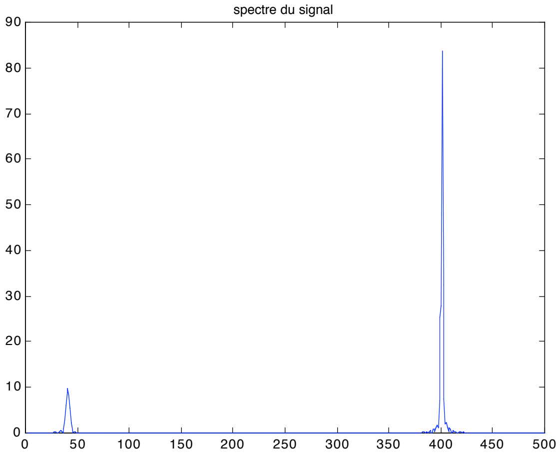
- Principal Components Analysis (PCA)
- Kahrunen-Loeve decomposition (continuous signals)
- Diagonalization of a covariance matrix
  - Eigenvectors give the principal directions of vector clusters
  - Eigenvalues give the variances along each principal direction
- Singular Spectrum Analysis

Time-frequency analysis

Problems with Fourier analysis

- No spatial information





- Not a realistic model (destructive interferences)

$$x(t)=\int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}\,df=0 \quad t\notin[0,T]$$

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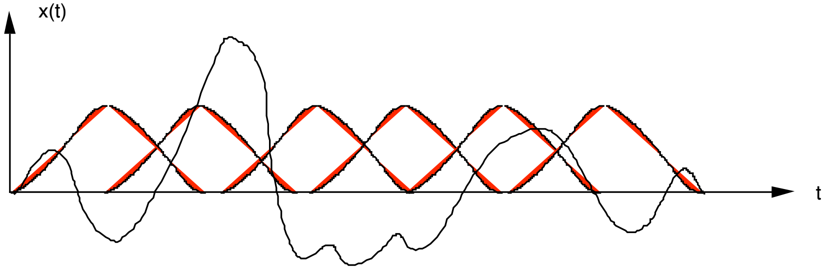
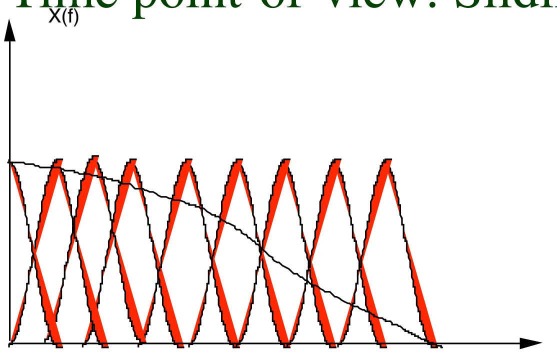
Time-frequency analysis

Short-Time Fourier transform



analog to a new transform with function g(s)

Time point-of-view: Sliding window



$$g_t(s) = w(s-t)e^{j2\pi fs}$$

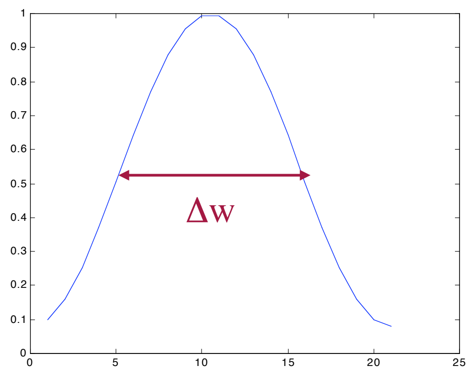
$$X(t,f) = \int_{-\infty}^{\infty} x(s)w^*(s-t)e^{-j2\pi fs}ds$$

$$X(t,f) = \int_{-\infty}^{\infty} x(s)g_t^*(s,f)ds$$

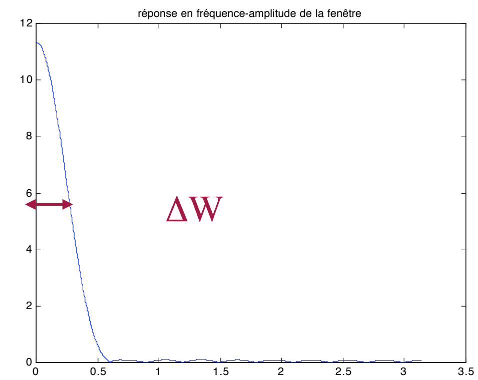
Frequency point-of-view:Bandpass filter bank

Time-frequency analysis

Uncertainty principle



$$\Delta w.\Delta W \geq \frac{1}{4\pi}$$



Time window

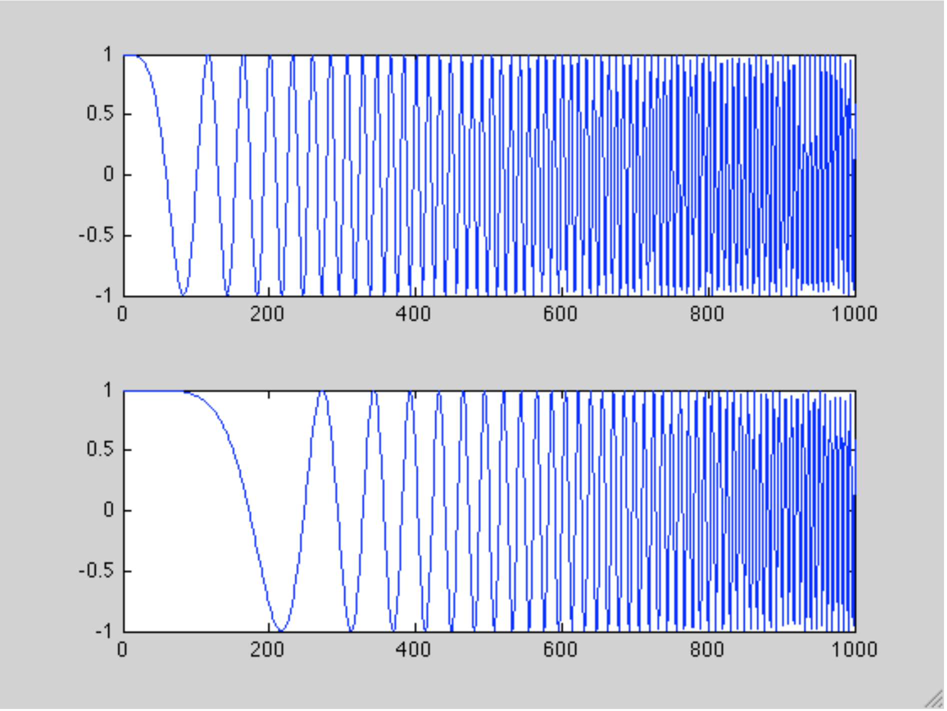
Window Spectrum

- Good time resolution, bad spectrum resolution
- Good spectrum resolution, bad time resolution

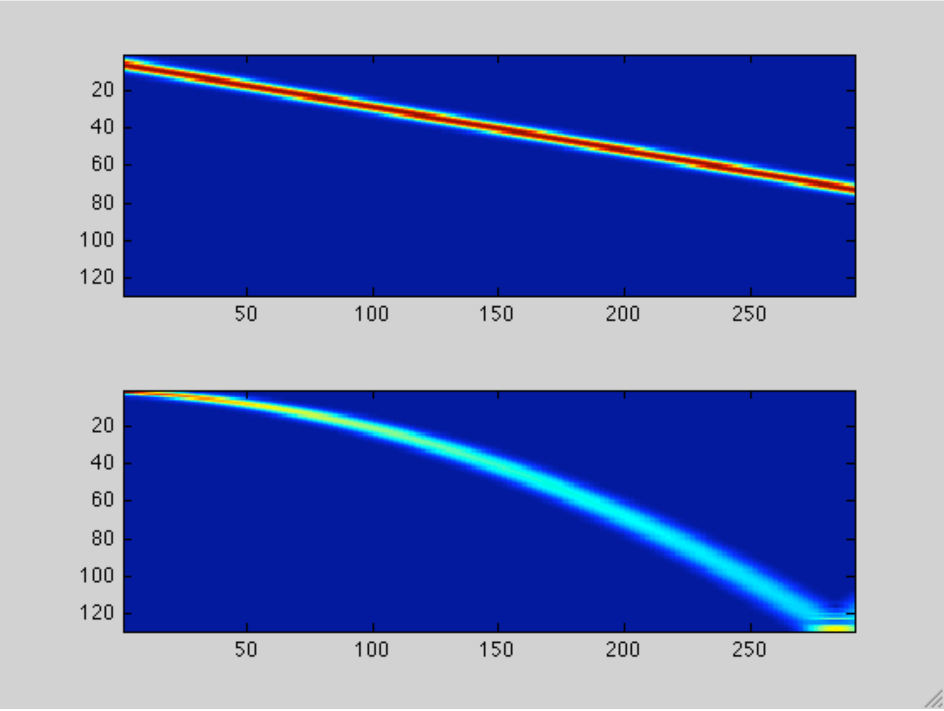
Time frequency analysis

- **Spectrogram (spectrum function of time)**

  Linear Chirp (up) and quadratic chirp (bottom)



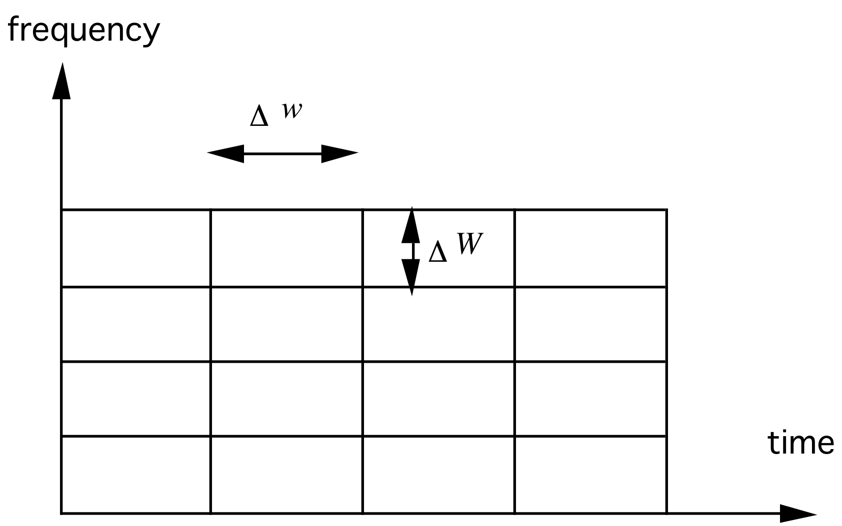
Signal

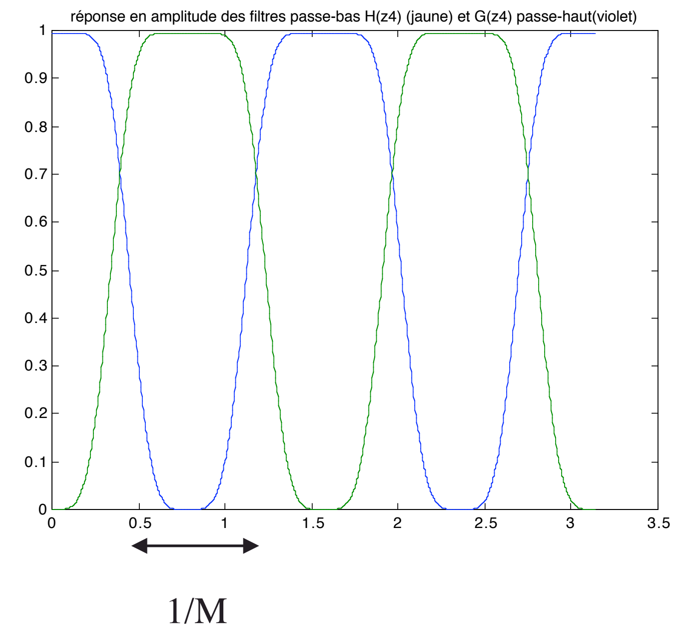


Spectrogram

Time frequency analysis analysis

Modulated filter bank



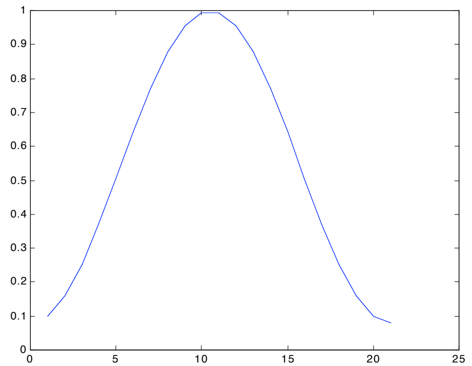


Time-frequency plane tiling

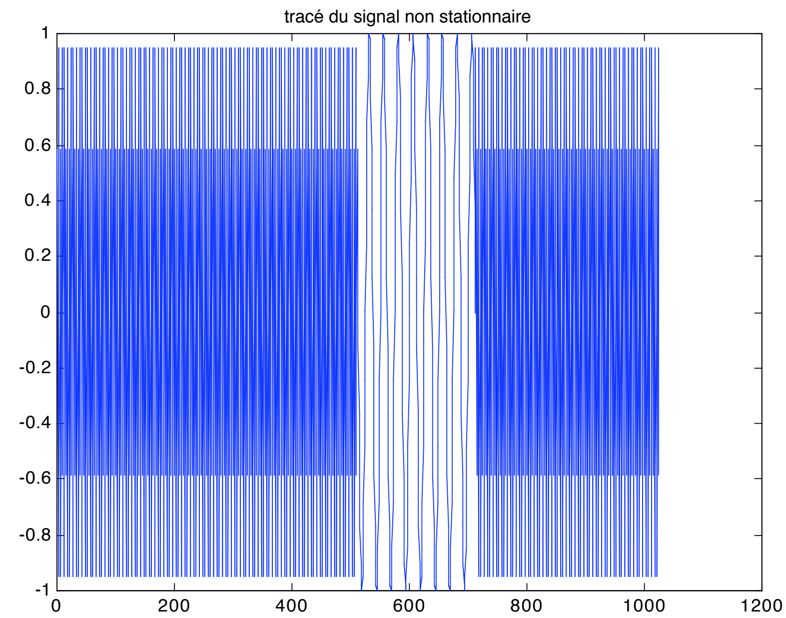
Frequency point-of-view: Modulated filterbank

Time-frequency analysis

Short-time Fourier Transform

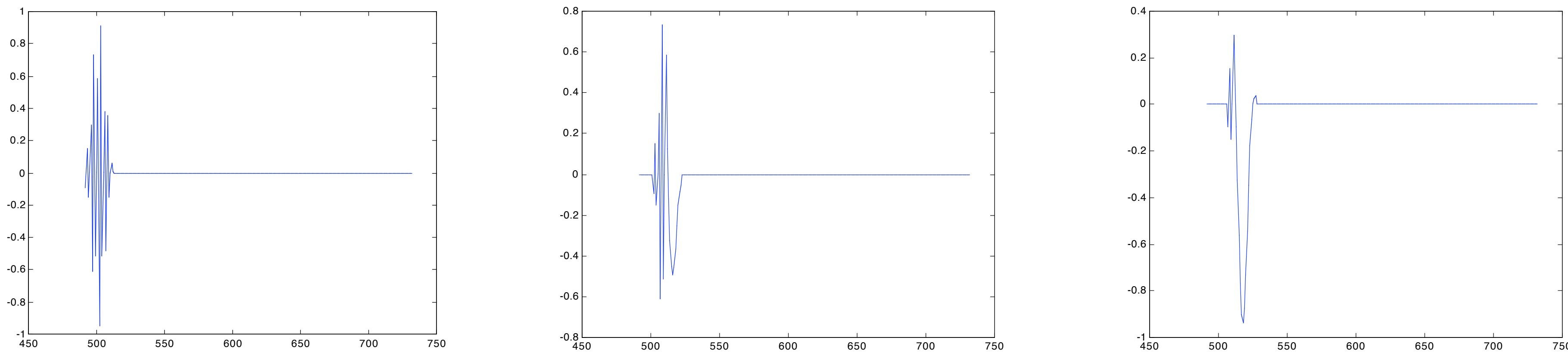


Hamming window

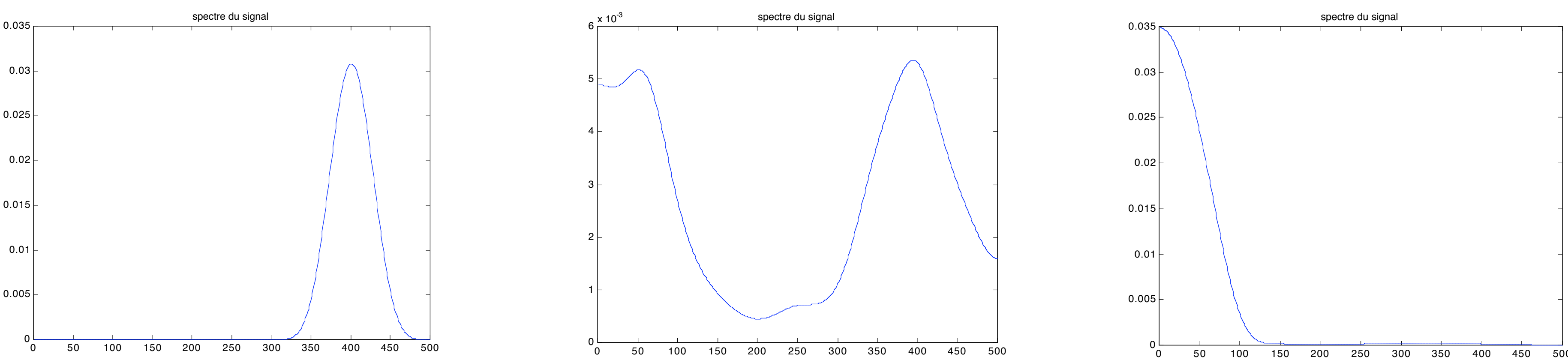


Non stationary signal

Time frequency analysis



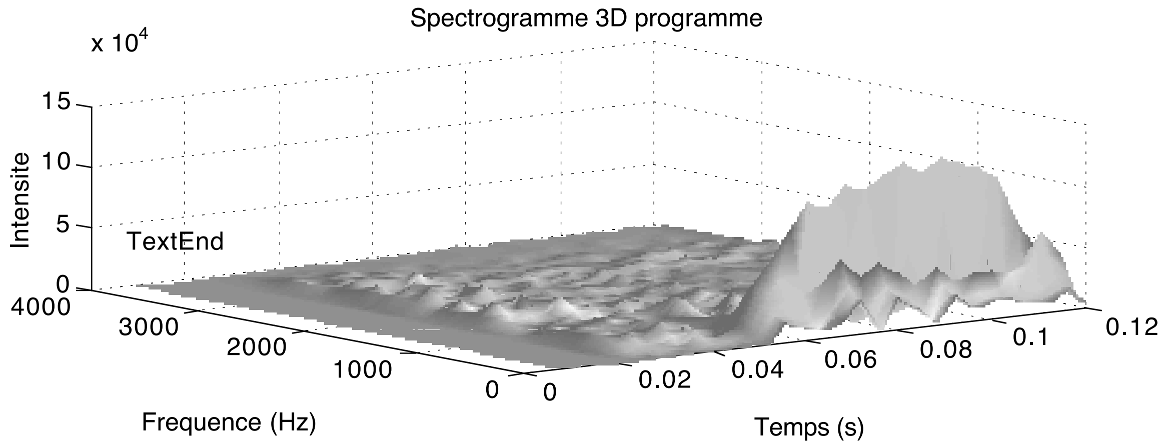
Signal

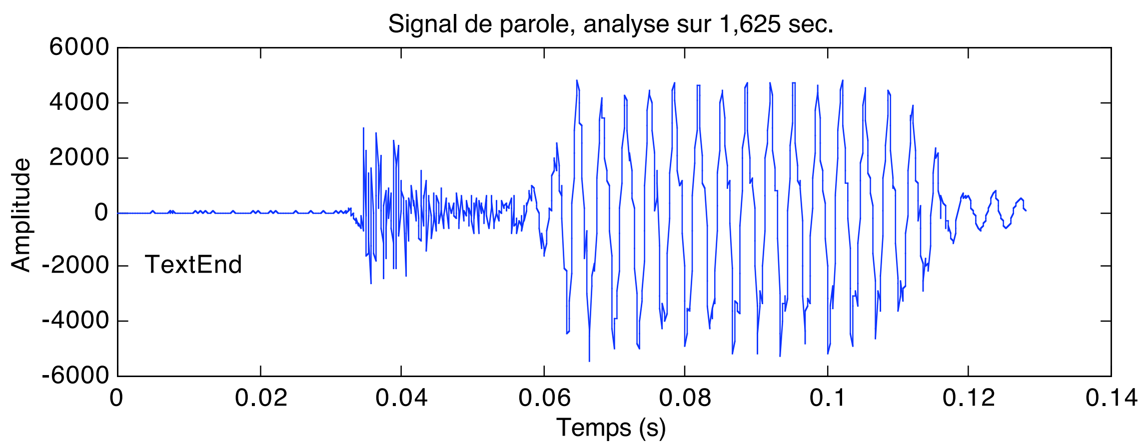


Spectrum

Time frequency analysis

Speech signal: spectrogram





Time-Scale analysis

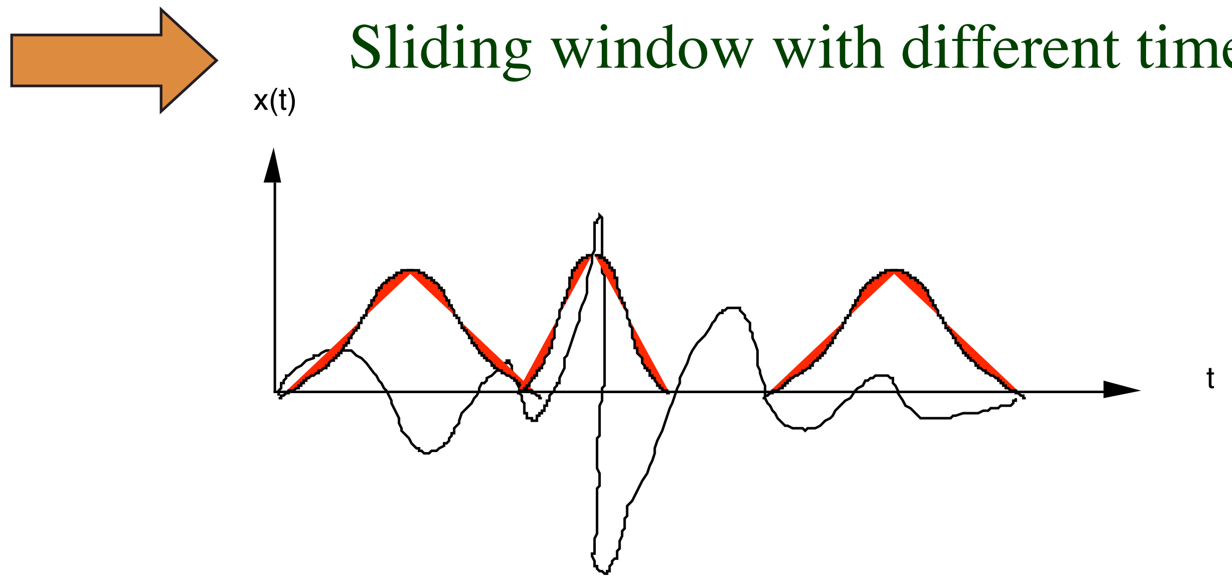
Statement

- The information, which is seeked in a signal, varies.

Transients, bursts, fast changes : fine time resolution

Periodic signals, long term evolution : fine frequency resolution

Sliding window with different time support



- **Large window for long term evolution , Short window for transients**

Continuous wavelet transform

$\psi(t)$ is a normalized window called « wavelet » with zero mean

$$\int \psi(t)dt = 0 \qquad \int |\psi(t)|^2 dt = 1$$

New transformation over dilated and translated wavelets

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) \qquad b \in R, a \succ 0$$

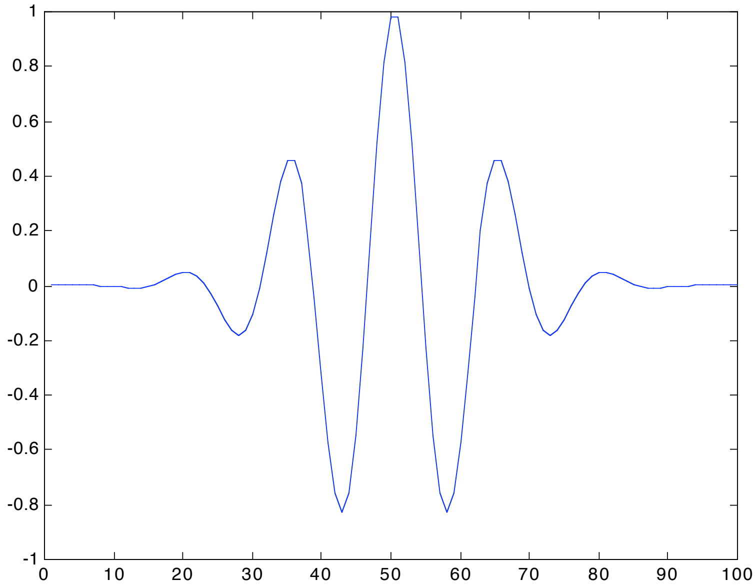
$$C_{a,b} = \int_{-\infty}^{\infty} x(t)\psi^{*}{}_{a,b}(t)dt$$

a: scale factor

b : translation factor

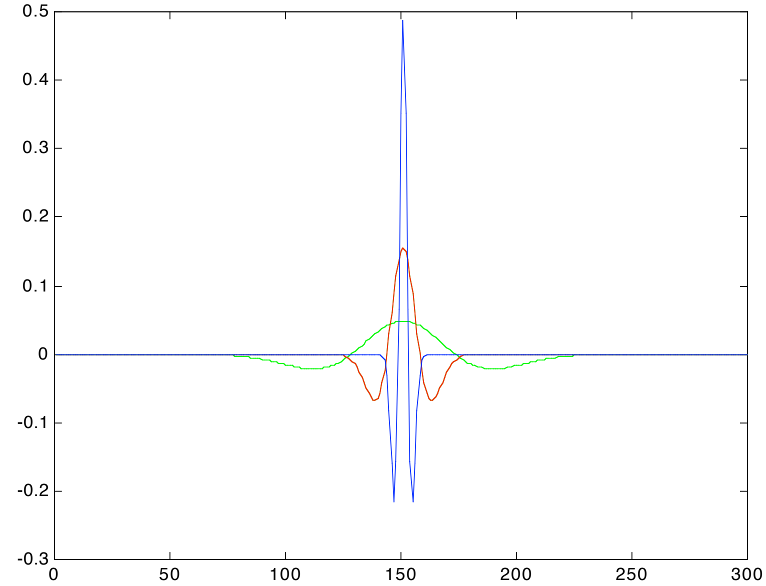
Different wavelets

- Morlet wavelet



$$\psi(t) = e^{-t^2/2}\cos(5t)$$

- Mexican hat



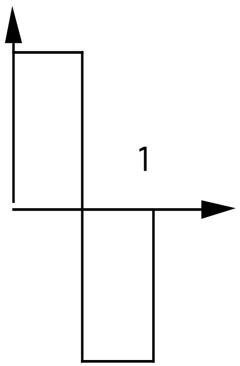
- a=0.1
- a=0.01
- a=0.001

- Second derivative of gaussian

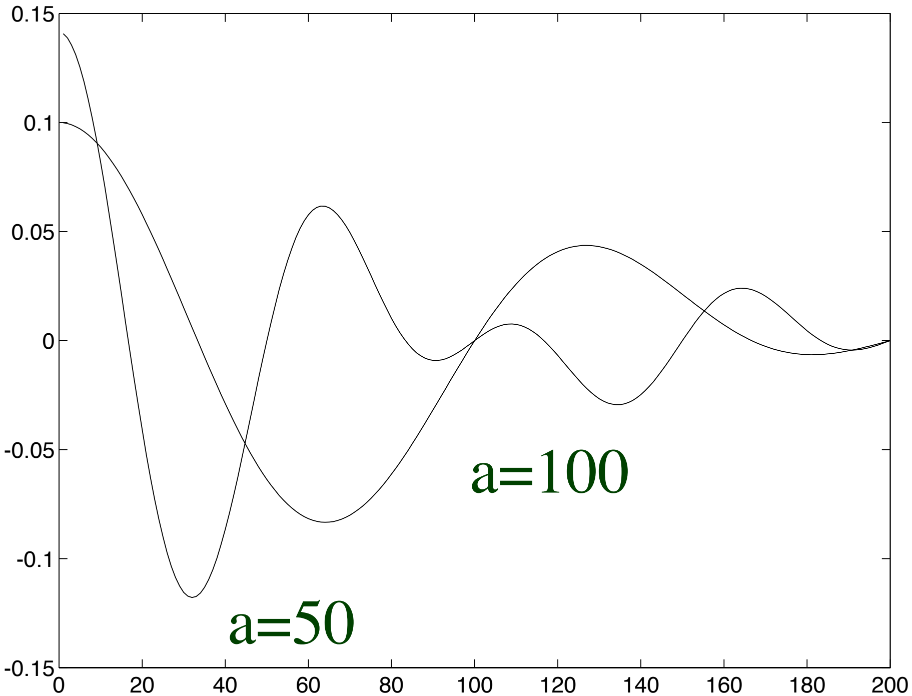
$$\psi(t) = \left(\frac{2}{\sqrt{3\sqrt{\pi}}}\right)\left(1-t^2\right)e^{-t^2/2}$$

Different wavelets

- Haar wavelet



- Shannon wavelet



Wavelet transform

- Inverse continuous wavelet transform

$$x(t) = \frac{1}{C_\psi} \iint C_{a,b} \psi_{a,b}(t) \frac{da\,db}{a^2}$$

Admissibility condition: analysis and reconstruction without error

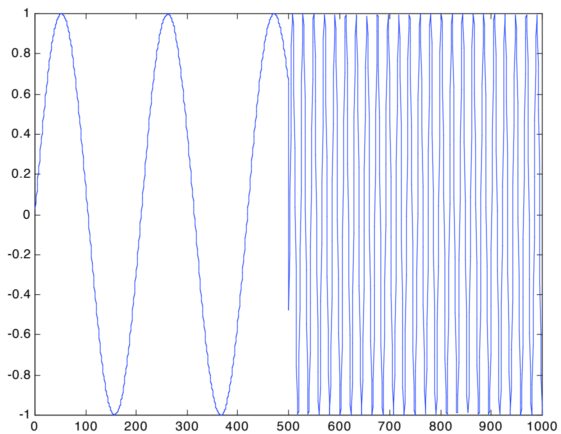
$$C_\psi = \int \frac{|\Psi(\omega)|^2}{|\omega|} d\omega \prec \infty \quad \Rightarrow \quad \Psi(0) = \int \psi(t)\,dt = 0$$

- A wavelet is a symetric time window with null mean

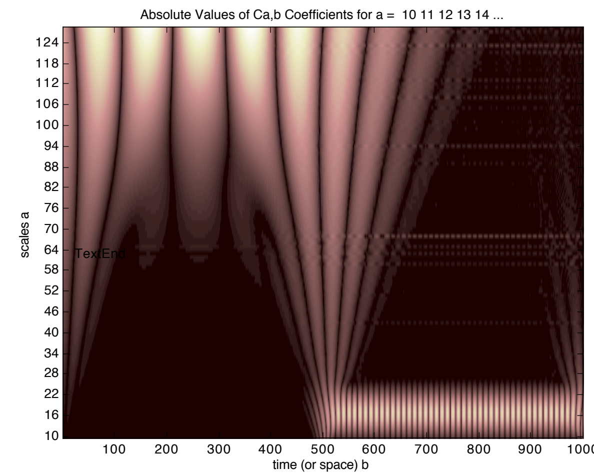
Example

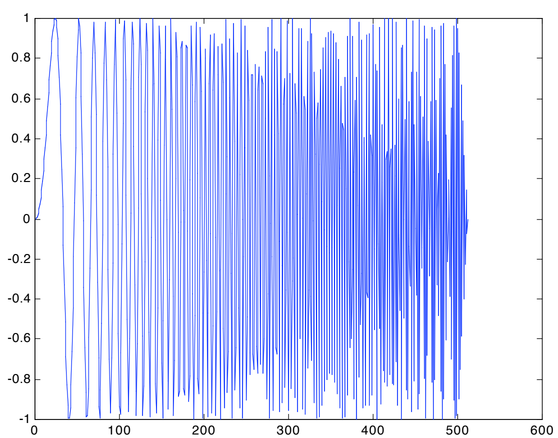
Signal

Scalogram

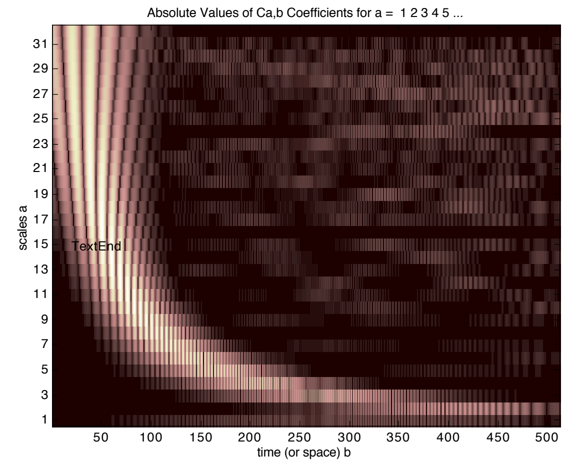


Breakdown of sinusoïds



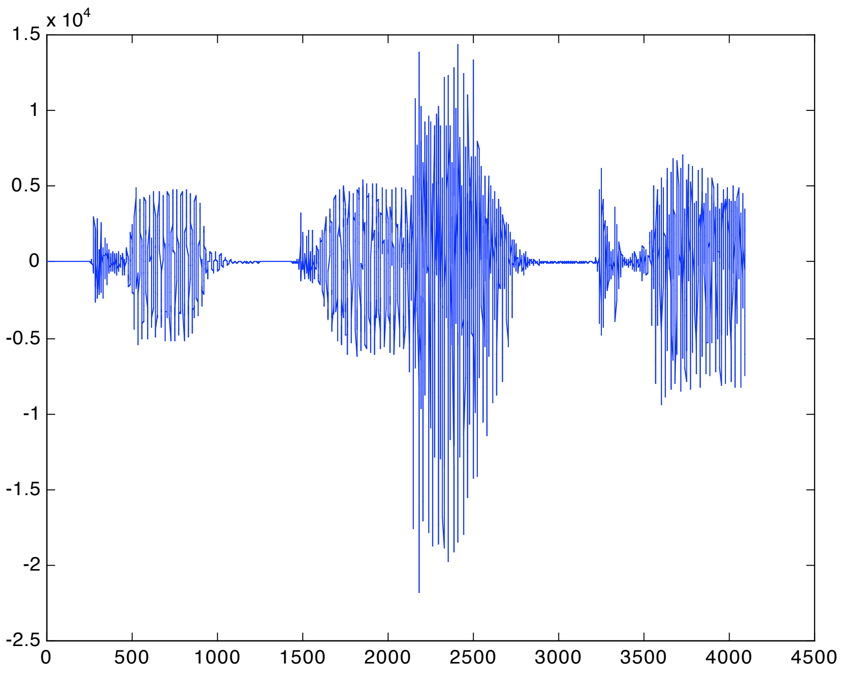


Linear chirp

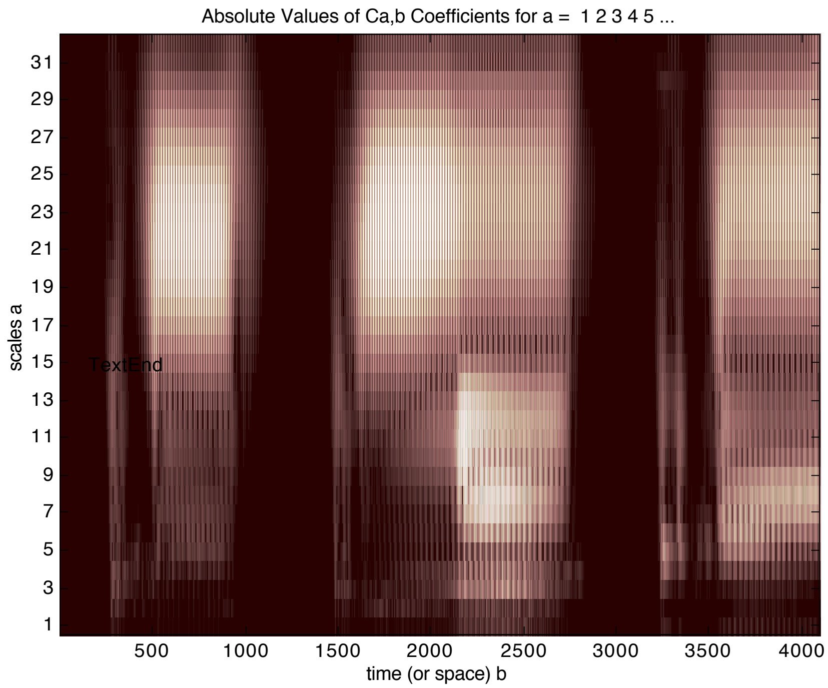


Example

Speech Signal



Scalogram (Morlet) a=1:32



a=1: high frequencies, short scales

a=32: low frequencies, large scales

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Expansion on orthonormal basis

- **Special choice of sampling in time and scale leads to orthonormal wavelets**

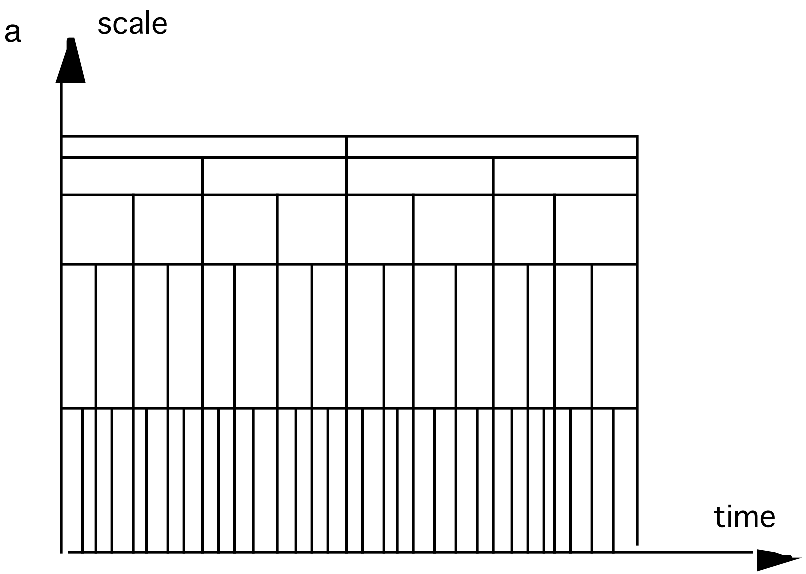
$$\int \psi_{j,k}(t)\psi_{n,m}(t)dt = \begin{cases} 1 & j = n, k = m \\ 0 & ailleurs \end{cases}$$

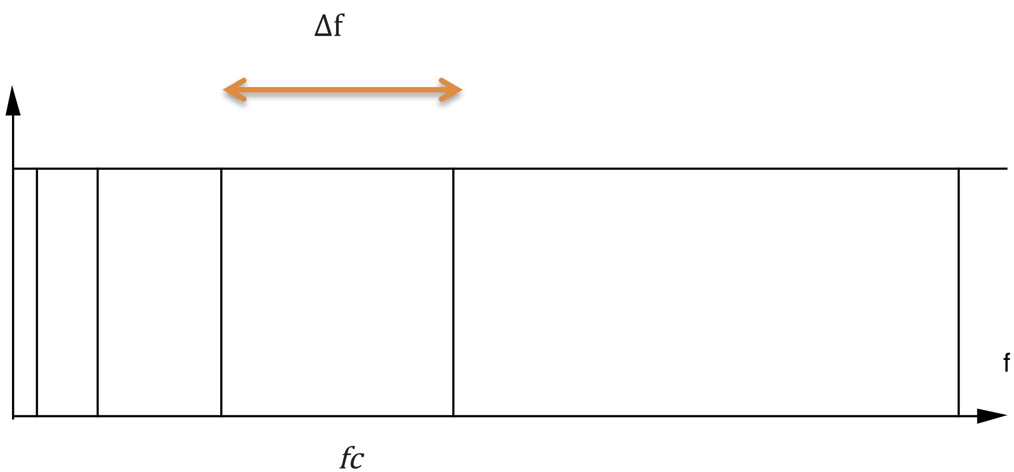
- This a the case of a dyadic sampling in scale (power of 2)
  - a=2^j
  - b=a.k

$$\Psi_{j,k}(t) = 2^{-j/2}\Psi(2^{-j}t - k)$$

→ Non redundant representation of signal

Time-scale analysis

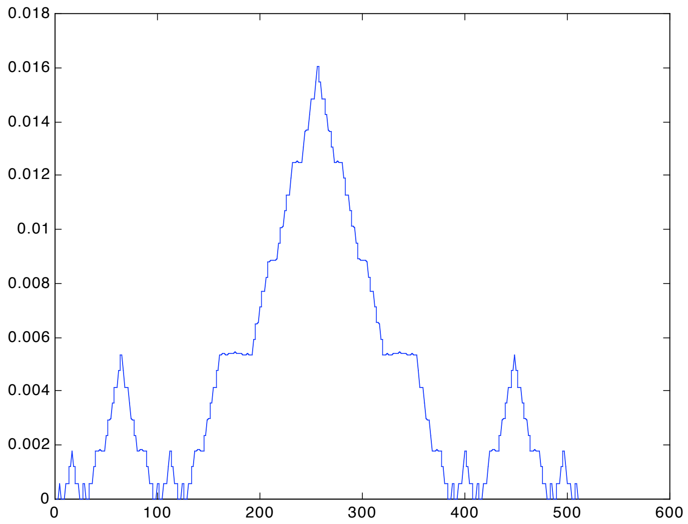




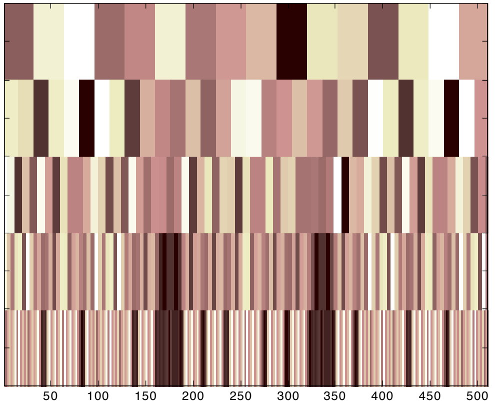
- Time-scale tiling
- Dyadic filter bank: $\frac{\Delta f}{f_c} = constant$

Time-scale analysis

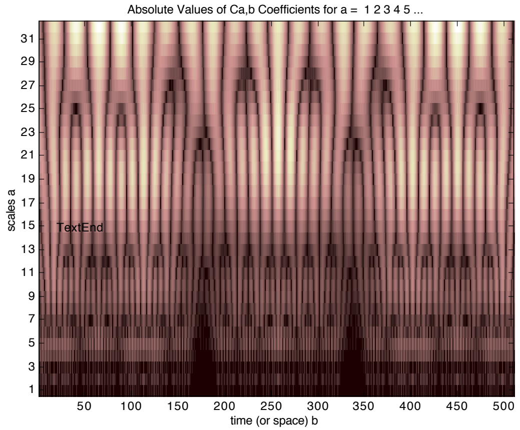
- **Fractal signal : Von Koch curve (flake)**



Discrete Scalogram 5 levels, Daubechies 4

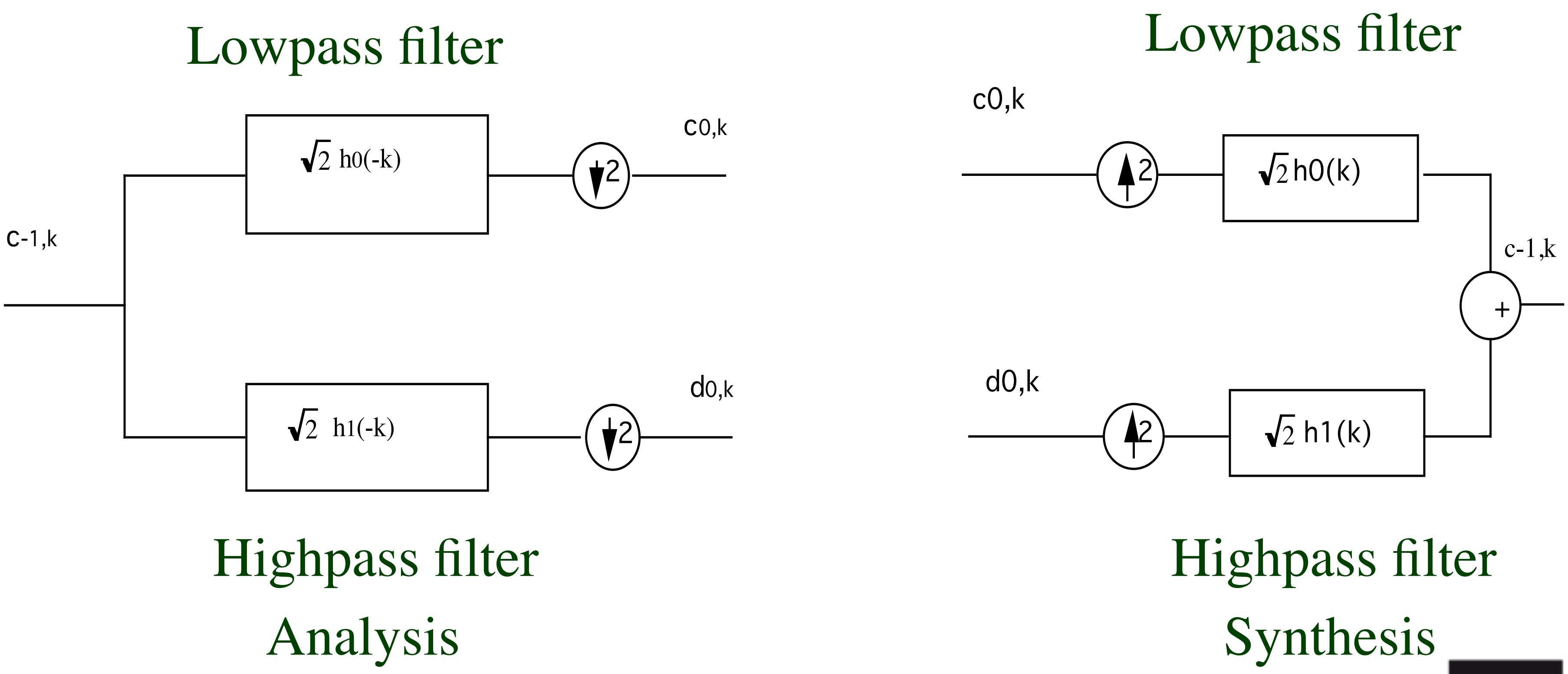


Continuous Scalogram, Morlet



Discrete wavelet implementation

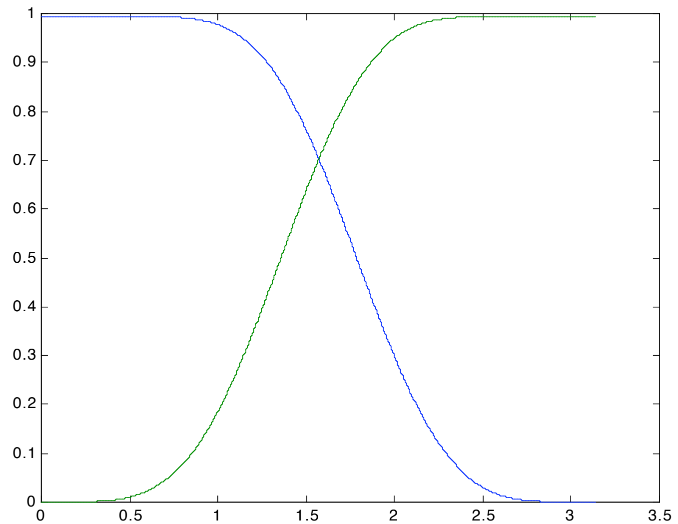
- **Decomposition and reconstruction of signal between scale 0 and -1 by Mallat's algorithm**
  - No need for wavelet function, only filter coefficients



Discrete wavelet implementation

Conjugate mirror filters (Daubechies filters)

The lowpass and highpass filters have symetrical transfer function for ω=π/2



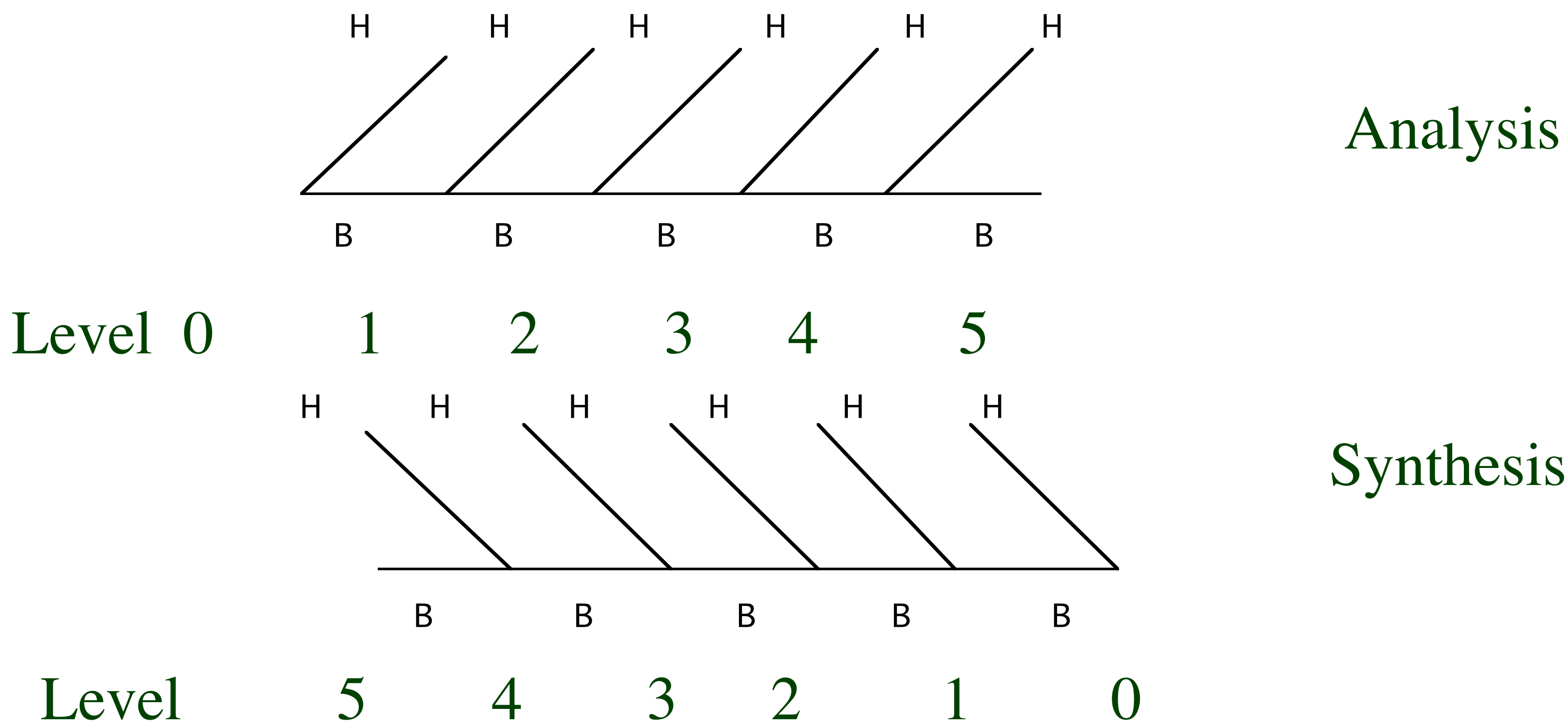
Condition of orthogonality for reconstruction without errors

$$\left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1$$

Discrete wavelet analysis by Mallat's algorithm

$$c_{0,k} = s(k)$$

B: lowpass filter(approximation)
H: highpass filter(details)



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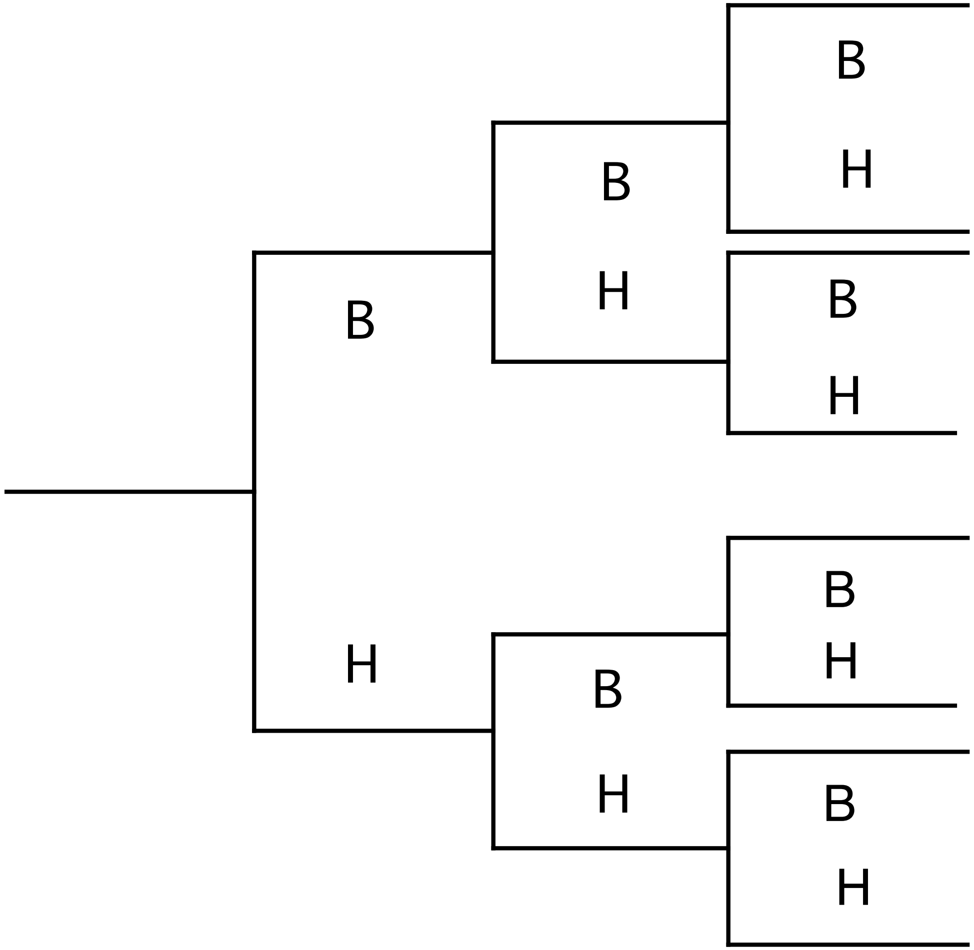
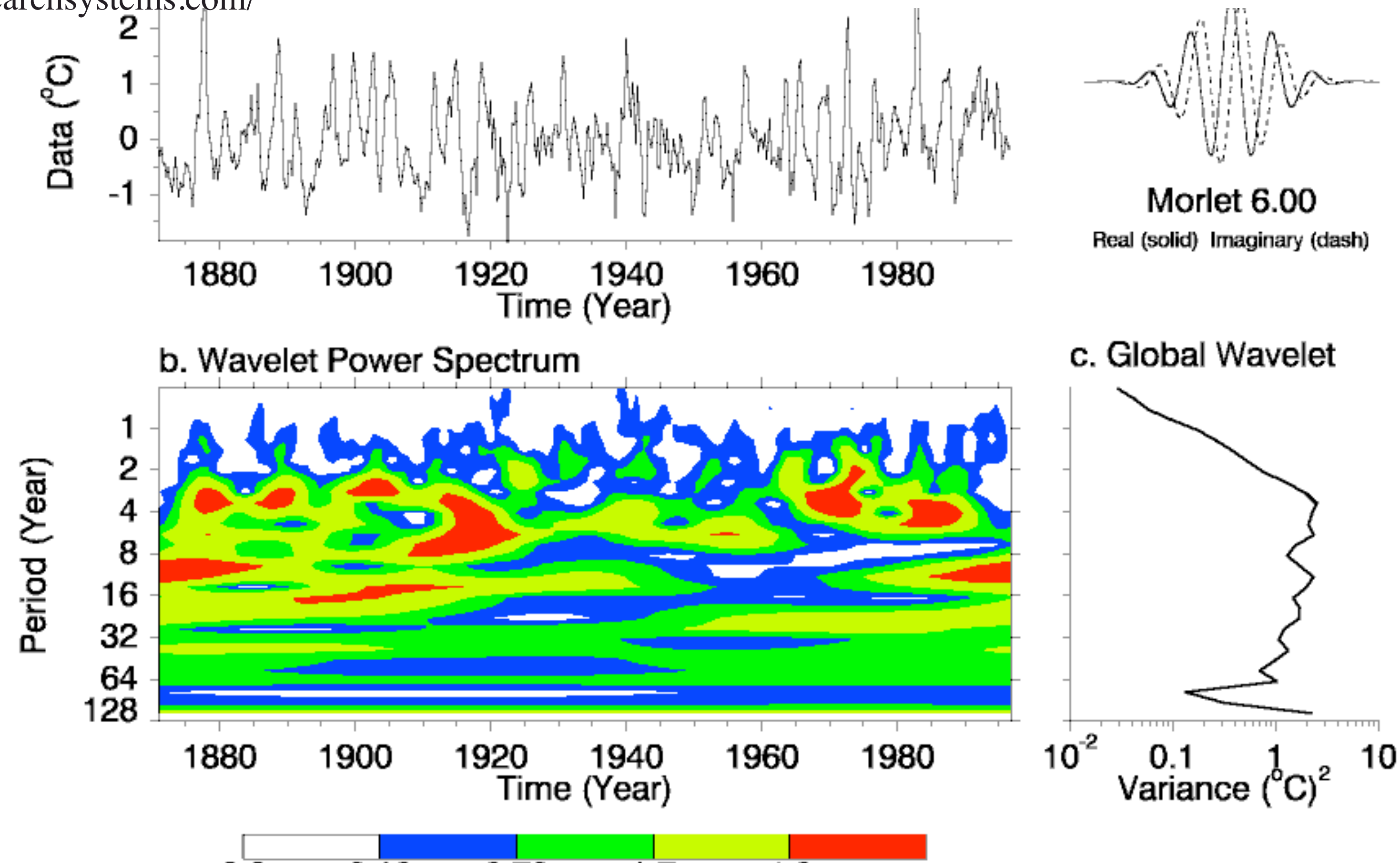
Wavelet packets



Analyse de la température de surface d'El Nino par ondelettes

http://ion.researchsystems.com/



(a) Data. (b) The wavelet power spectrum. The contour levels are chosen so that 75%, 50%, 25%, and 5% of the wavelet power is above each level, respectively. (c) The global wavelet power spectrum. Reference: Torrence, C. and G. P. Compo, 1998: A Practical Guide to Wavelet Analysis. Bull. Amer. Meteor. Soc., 79, 61-78.

Wigner-Ville transform

- **Search for a transform with following properties**

Instantaneous density $T_x(t,f) \longrightarrow$ Energy $E = \int\int T_x(t,f)dtdf$

Marginal densities

$$\int_{-\infty}^{\infty} T_x(t,f)dt = |X(f)|^2$$

$$\int_{-\infty}^{\infty} T_x(t,f)df = |x(t)|^2$$

- **Choice of a transform with these properties**

Instantaneous energy density: $R(t,\tau) = x\left(t+\frac{\tau}{2}\right)x^*\left(t-\frac{\tau}{2}\right)$

Wigner-Ville transform: $W_x(t,f) = \int_{-\infty}^{\infty} R(t,\tau)e^{-j2\pi f\tau}d\tau = \mathcal{F}_\tau\left(R(t,\tau)\right)$

Wigner-Ville transform

Bilinear transform $W_{x+y}(t,f) = W_x(t,f) + W_y(t,f) + 2\,\mathcal{R}e(W_{xy}(t,f))$

Problem: interference terms without meaning

$$W_{xy}(t,f) = \int_{-\infty}^{\infty} x(t+\frac{\tau}{2})y^*(t-\frac{\tau}{2})e^{-j2\pi f\tau}\,d\tau$$

Improvement : filtering in the spectrum plane (or equivalent windowing in the time plane)

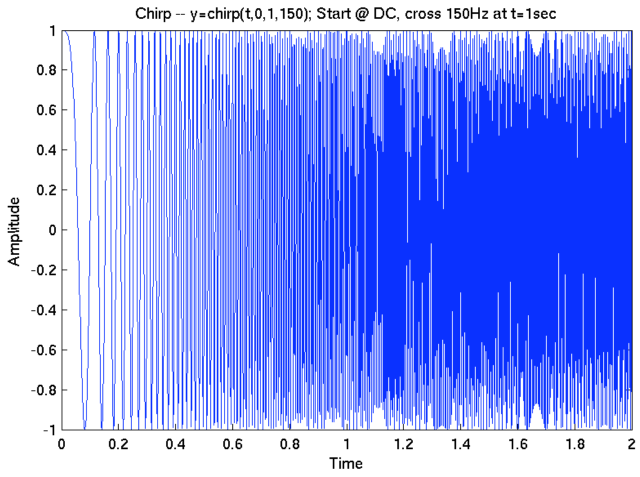
Pseudo Wigner-Ville $PW_x(t,f) = \int_{-\infty} p(\tau)x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-j2\pi f\tau}\,d\tau$

→ less interferences, but less resolution

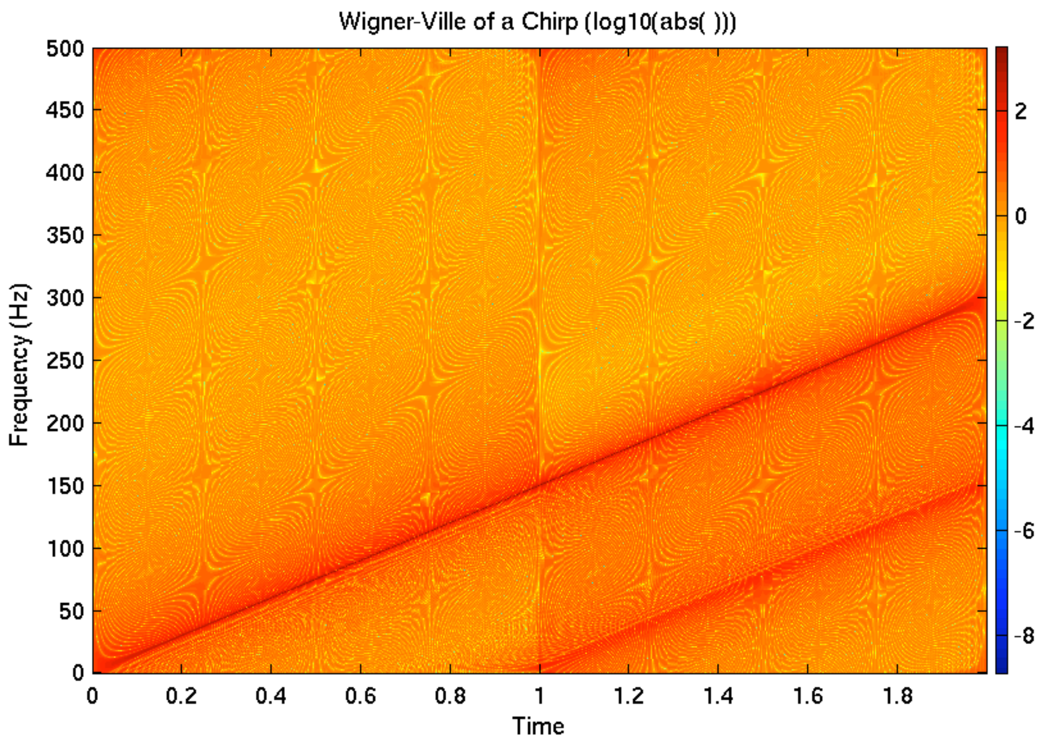
- Good properties for spectral analysis of frequency modulated signals (dolphins, whales, bat,…)

Wigner-Ville transform

- **Example : chirp**



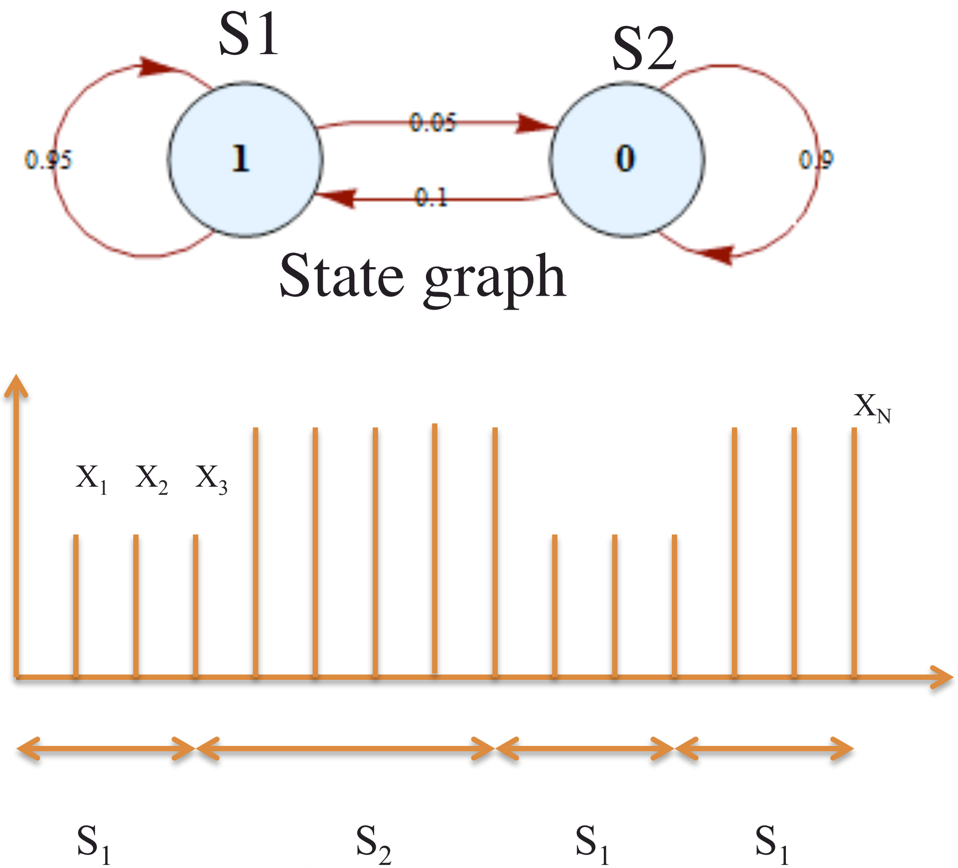
Linear frequency modulation of a sine wave



Wigner-Ville transform

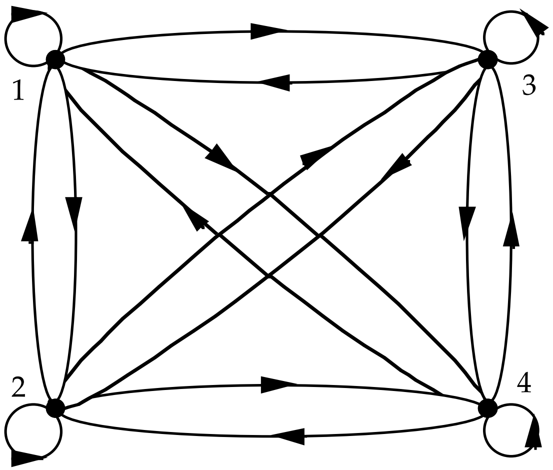
Hidden Markov models

- **Markov chain : sequence of states driven by transition probabilities** (example of 2 states)



Hidden Markov models

- **Markov chain : fully connected (example of 4 states)**



$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$et, \forall (i,j), a_{ij} > 0$$

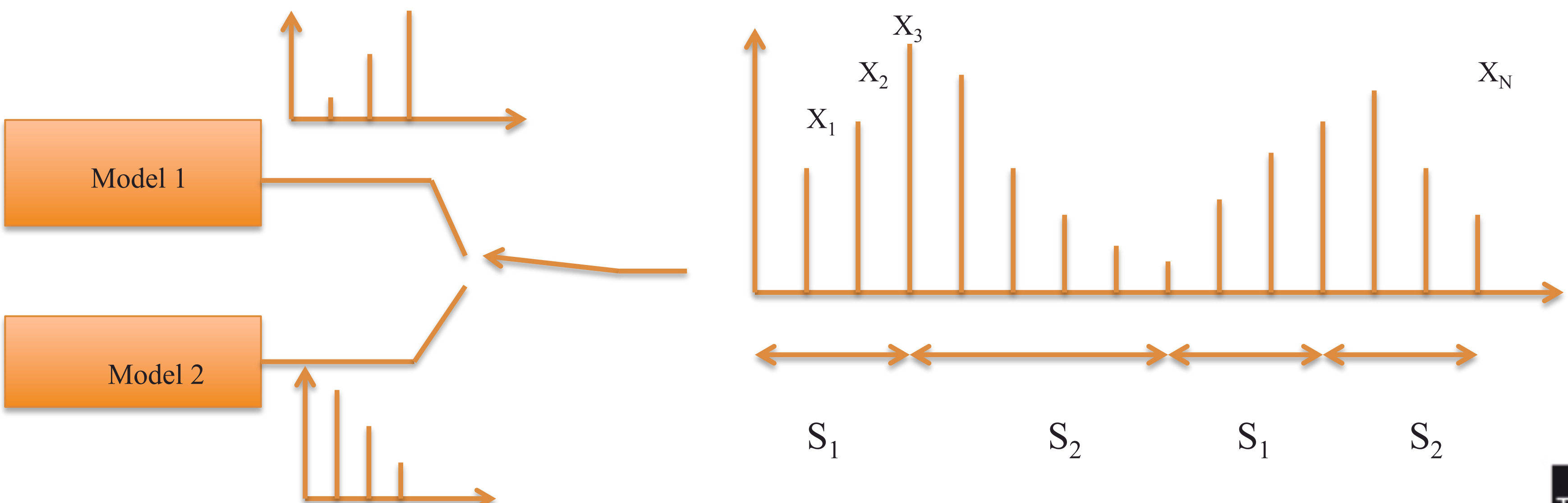
State transition matrix

- **Markov chain description (M states)**
  - State set **s**=(S_i, i=1…M) where Si means state i.
  - Transition matrix **A** = (a_{ij} / i=1,...,M ; j=1,...,M) where a_{ij} is the probability of going from S_i at time t-1 to state S_j at time t.
  - Initial state probabilities $\pi = (\pi_i$ / i=1,...,M), where π_i is the probability of being in state S_i at time t=0.

Hidden Markov models

Hidden Markov chain

- No direct observation of states, but sequences of observations
- Description by (**s,A,p,**π)
  - Markov chain description (**s,A,** π)
  - Probability distribution of observations in each state : **p**= ($p_i(x)$ / i=1,...,M) where $p_j(x)$ is the probability density of observation x conditionally to state i.

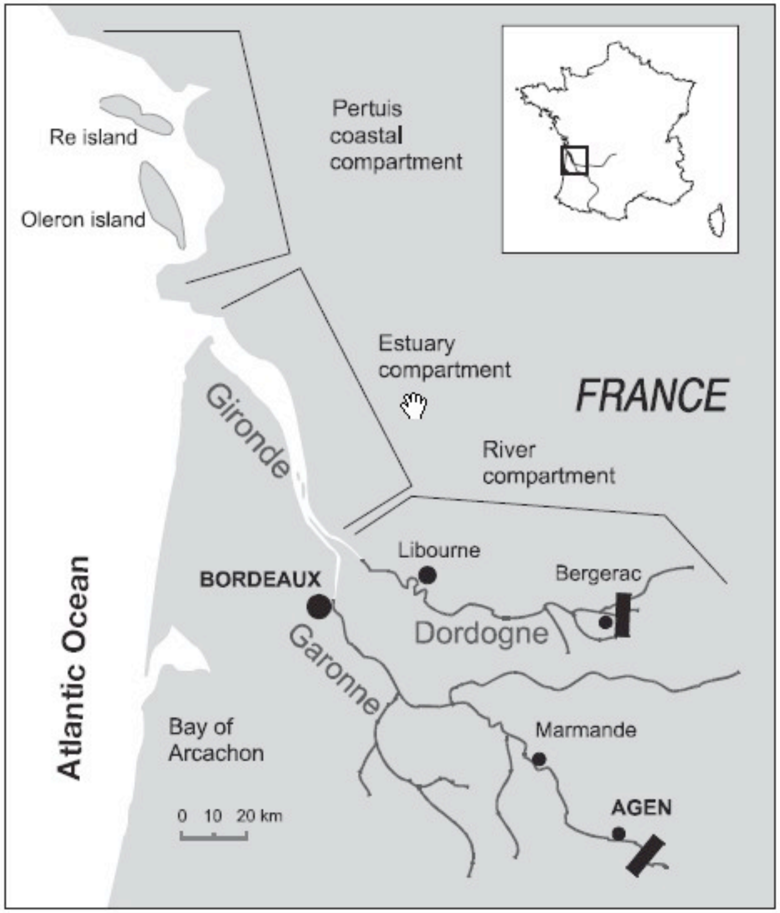


Hidden Markov models

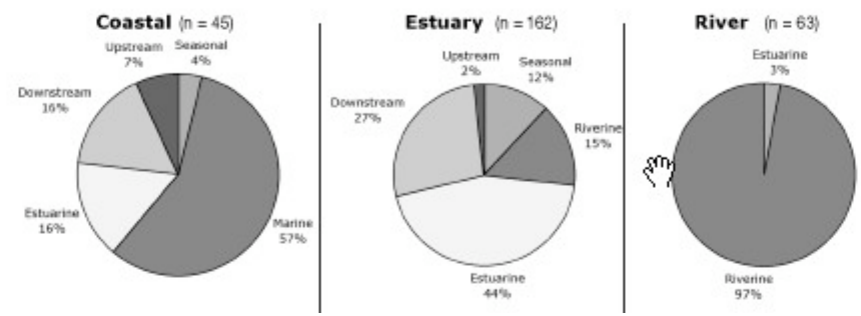
Problems to be solved

- Pattern recognition problem: Choice of the best model to describe time series (each model being associated to a pattern)
- Solution: maximization of the probability distribution $p(\mathbf{x}_{1:N}/\lambda_\iota)$ where $\mathbf{x}_{1:N}$ is the observation sequence $x_1,x_2,\ldots\ldots,x_N$ and $\lambda_\iota = (A_i,\mathbf{p}_i,\pi)$ is the model i with i=1,…,M
- Classification problem: recovering of the state sequence $\mathbf{s}_{1:N}$ from the noisy observation sequence $\mathbf{x}_{1:N}$
- Solution: maximization of the probability distribution $p(\mathbf{s}_{1:N}/\mathbf{x}_{1:N},\lambda)$
- ⇒ Viterbi algorithm (or dynamic programming)
- Estimation problem : estimation of the model parameters
- Solution : maximization of the likelihood function $p(\mathbf{x}_{1:N}/\lambda)$
- ⇒ Baum-Welch algorithm

Migration of the eels

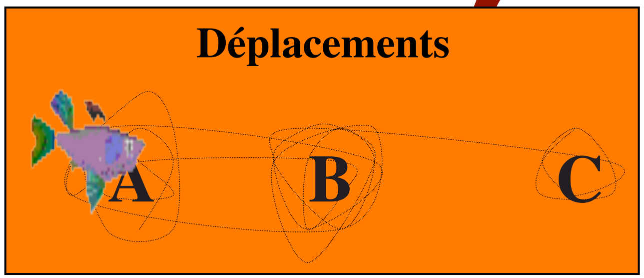


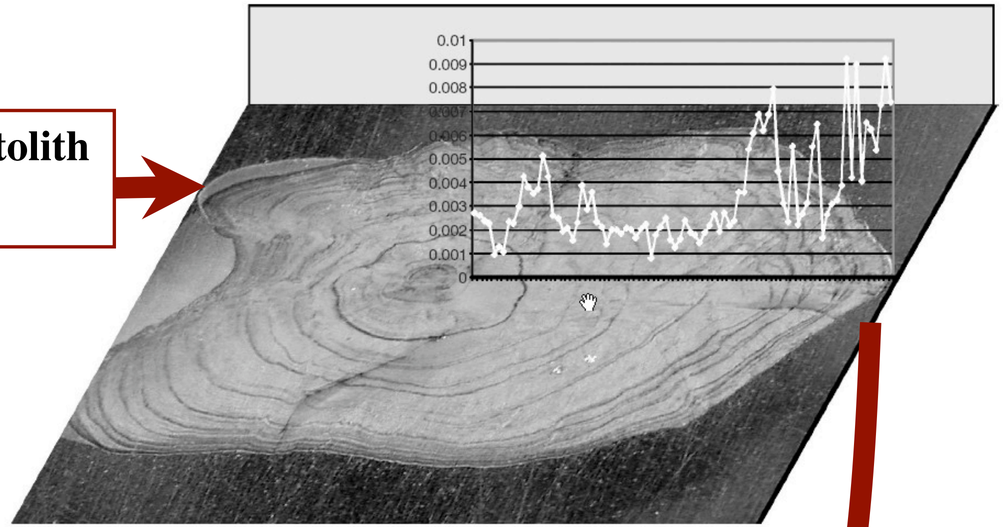
- Correlation between the ratio Strontium/Calcium of otoliths and migration areas (coastal, estuary, river)



Migration of the eels

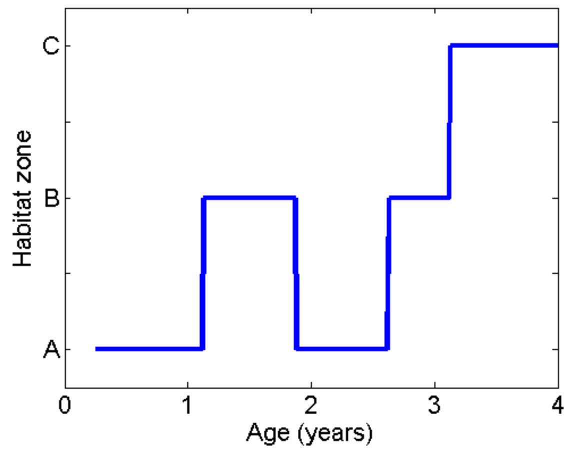
Signal from otolith analysis





Hidden states : migration areas

How to find the hidden states ?



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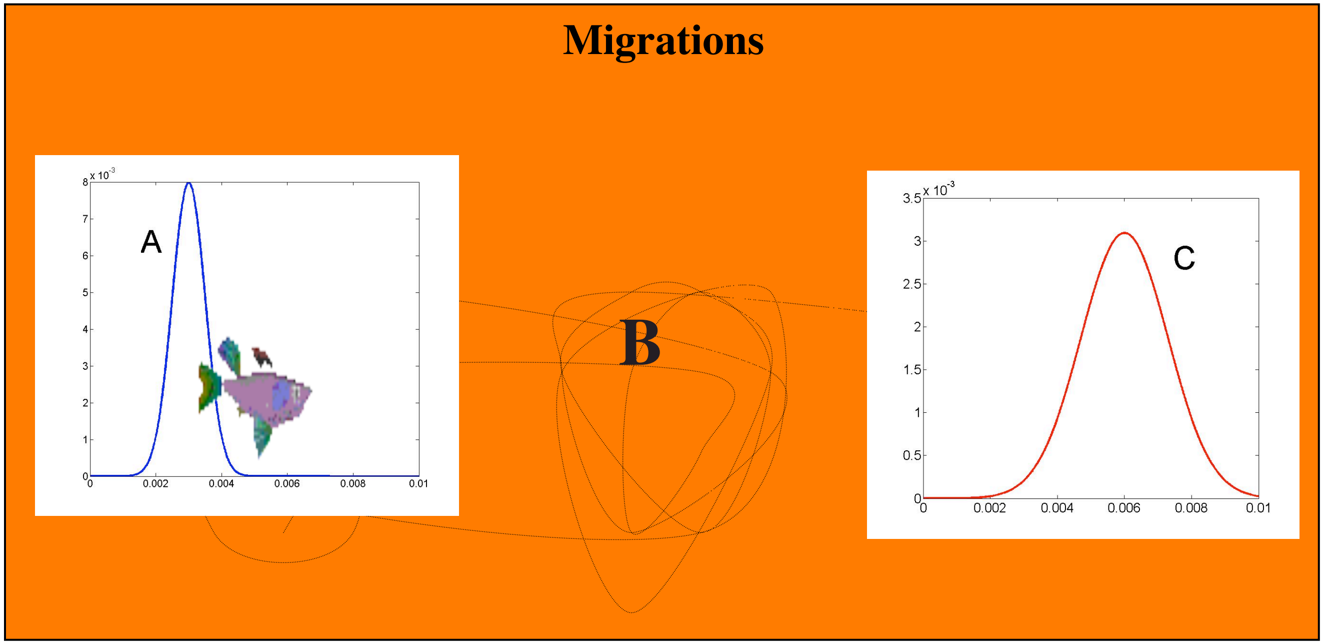
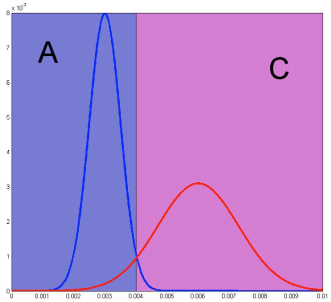
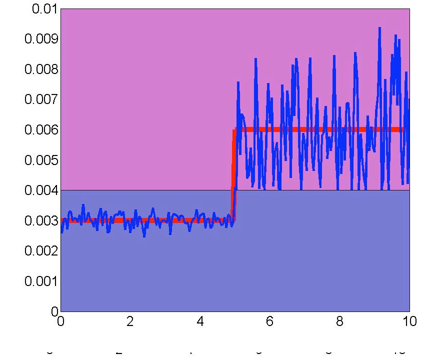
Migration of the eels

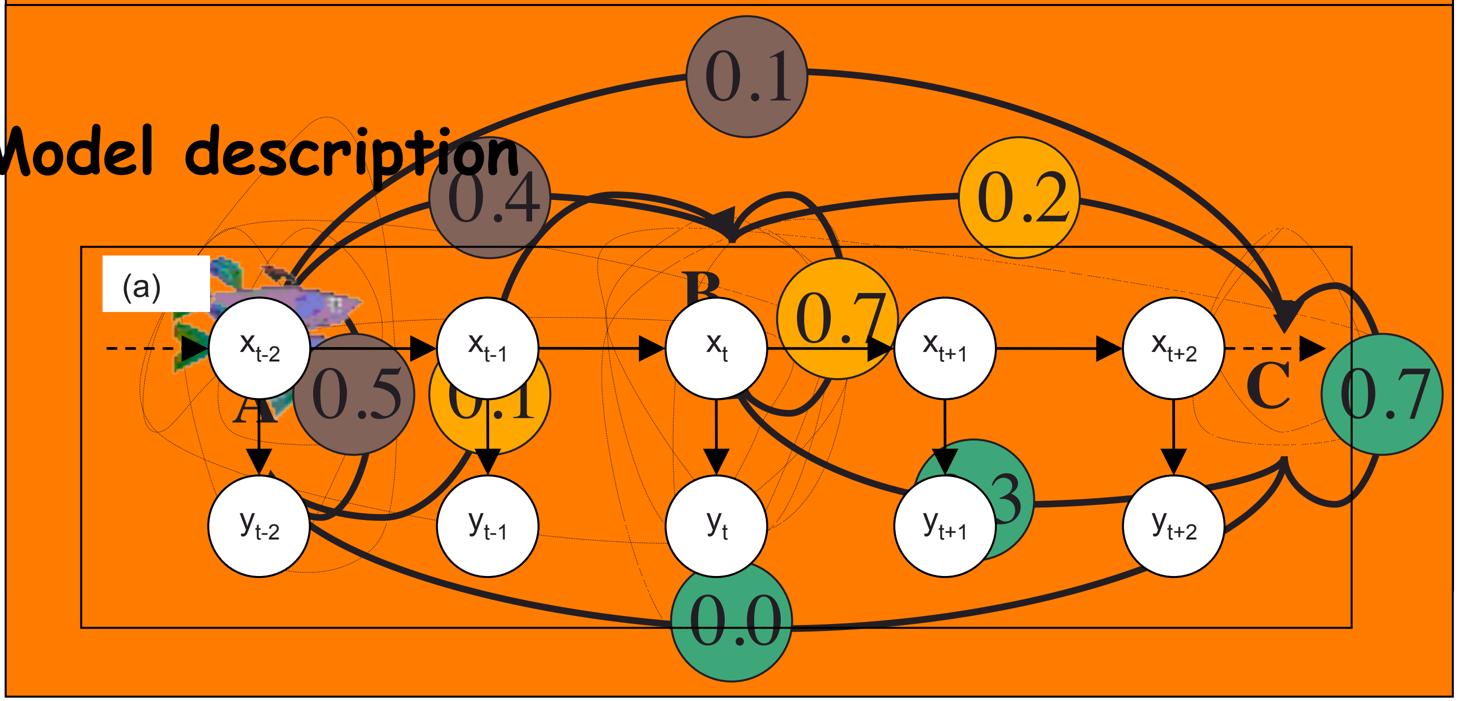






Migration of the eels

- **HMM model characteristics**
  - 1 :probability distributions of the observations for each state
  - 2 : transition dynamic between states
- **Model description**



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